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Nikolaí N. Nekhoroshev

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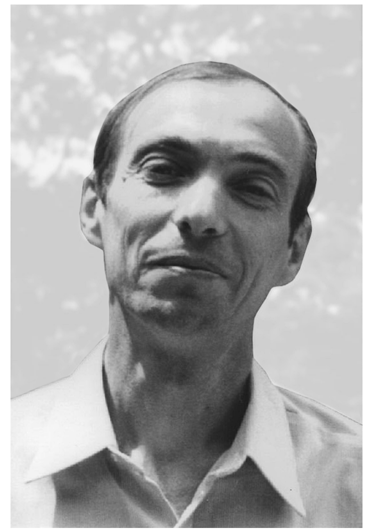
In 2016 we celebrate the 70-th birthday of Nikolaí Nekhoroshev, a prominent Russian mathematician who is widely known for his theory of exponential stability of nearly integrable Hamiltonian systems. The present issue of the journal *Regular and Chaotic Dynamics* is dedicated to his memory.

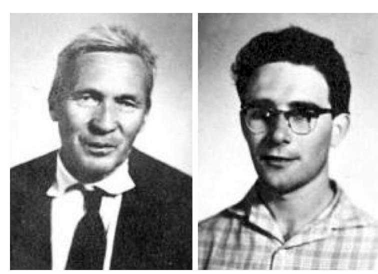


Nikolaí Nekhoroshev, 1946–2008.

Nikolaí Nekhoroshev was born on October 2, 1946, in Kursk, a town in central Russia (then USSR), some 500 kilometers south of Moscow. His father (1902–1984) served as a border guard, fought in the war of 1941–1945, and after that he worked as a veterinary paramedic. His mother (1905–1991) was a homemaker, and his three sisters Greta (1929), Eugenia (1934), and Lyudmila (1939) continued to live in Kursk in the 2000s. The older sister Greta recalls that Nikolaí lived in Kursk with his parents, while the sisters had separate homes, and Greta lived in the Far East. Being the youngest child, and the only and long-awaited son, the boy was his mother's favorite. She was a high school graduate and took a very active part in his life, went to teacher–parent meetings, and helped with his homework. It is likely that the features of his personality such as tact, self-discipline, and meticulousness in work were formed in his young years under her influence. Subsequently, Nikolaí remained very close to his mother and deeply mourned when she passed away. He never came back to Kursk after her death in 1991.

Very conscientious by nature, Nikolaí excelled in thoroughly preparing his homework. His keen interest in mathematics was awakened by his teachers whom he always remembered with gratitude. G. F. Dyominova (Leshchenko), a contemporary of N. N. Nekhoroshev and also Kursk-born, recalls how evening classes in physics and mathematics were organized in the Kursk teacher training college in the early 1960s. The professors of the college gave weekly courses on topics which were the most advanced at the time. The participants were learning a great deal. The courses were free and almost all senior high school students of Kursk who liked physics and mathematics were attending them. Usually about 100 students assembled in the lecture-hall and Kolya Nekhoroshev was among them.

At the same time, city and regional competitions in physics and mathematics for senior high school students were also held at the Kursk teacher's college. In February 1963, Nikolaí, then a tenth-grade student of the Kursk polytechnic high school No 38 with professional training, participated in the 4th mathematics competition for senior high school students of Kursk and the Kursk region and won the first place. He was sent to the all-Russian mathematics competition for senior high school students as a member of a four-student team from Kursk. It was during that trip that Galina Dyominova met him. She remembers Kolya as a serious and taciturn young man.



A. N. Kolmogorov and V. I. Arnold in 1964.

It should be noted that the organization of and preparation for all these events relied largely on the enthusiasm of physics and mathematics teachers who worked long hours with senior high school students after school, often without any compensation. Their selfless work and devotion paved the way into great mathematics for such gifted pupils as Kolya Nekhoroshev. We would like to take this opportunity to thank them.

According to the results of the all-Russian mathematics competition, where Nikolaí won a prize, 46 students, among whom he and Galina represented Kursk, were invited to participate in the first mathematics summer school in Krasnovidovo organized by academicians A. N. Kolmogorov and P. S. Aleksandrov. V. I. Arnold, a young doctor of science, the future scientific adviser of Nikolaí, was one of the lecturers at that school. The 1963 summer school gave impetus to establishing specialized high schools with extended mathematics curricula in the USSR. Many memories of this remarkable event are still vivid. The lecturers behaved in a very simple and friendly manner. It was inspiring to interact so closely with world class scientists. Galina Dyominova recalls that the undergraduate and postgraduate students were both teaching mathematics and counseling. They organized many intellectual games and activities, which were enjoyed by most students. Nikolaí, however, was not much interested in such divertissements. Instead, he continued to progress in his studies of mathematics. In September of the same year, he won the 3rd prize in the 2nd physics and mathematics competition of the European part of the USSR and Transcaucasia, which was organized by the staff of the Moscow State University (MSU).

First graduating class of the boarding school No 18 in 1964, Kolya Nekhoroshev is the 9th from the right.

V. I. Arnold with students of the boarding school No. 18. From left to right: V. I. Arnold, Andrei Kukushkin, Sasha Krygin, Kolya Nekhoroshev (in front), Valera Balin. The daily "Soviet Russia" of March 7, 1964. Photograph by N. S. Safonov.

At that time, A. N. Kolmogorov organized the boarding school No. 18 in Moscow with extended curriculum in physics and mathematics, known today as A. N. Kolmogorov Specialized Research and Training School associated with MSU[1),]where gifted students from the whole country were selected. Enrolments were held simultaneously for grades 9, 10, and 11. Enrolments for grade 11 were held in Krasnovidovo. In order to be admitted, the students had to pass examinations in two of the special courses which they had attended at the summer school. According to the results of these tests, 19 students including Nikolaí were admitted, see the photograph. Thus, at the age of 17 he left his native Kursk.

A. N. Kolmogorov and V. I. Arnold were among the teachers in the boarding school. V. I. Arnold, whom the students called simply Dima, gave particularly much attention to the graduating class. He gave a special course in ordinary differential equations. In the photograph, V. I. Arnold explains to his students how to construct phase portraits of one-degree-of-freedom systems using an example of particle motion in a double-well potential.

In 1964, now a graduate of the boarding school No 18 with extended physics and mathematics curriculum, N. N. Nekhoroshev was admitted to the Department of Mechanics and Mathematics of Moscow State University (MSU), known as "Mekhmat". From this moment until the end of his life he stayed attached to this department.

Starting in 1965, Nikolaí was an active participant and organizer of student summer construction brigades with which he traveled several times to Kazakhstan and Sakhalin. He was among the founders and in 1967 the first leader of Mekhmat's most famous student brigade — the Tyn Republic (*tyn* is the Kazakh for virgin land). Nikolaí Nikolaevich always recalled this time as one of the most remarkable periods of his life.

[1)] For more information consult `http://internat.msu.ru` and `http://www.pms.ru`

At the Department of Mechanics and Mathematics, N. N. Nekhoroshev began to participate in the seminar of V. I. Arnold. In 1966, V. I. Arnold, who was then a Mekhmat professor at the young age of 29, selected his first group of students, which included N. N. Nekhoroshev S. Zdravkovska, B. M. Ivlev, A. S. Pyartli, and A. G. K̇hovanskiĭ. The student years were rich in intense interactions with the scientific advisor, in many long walks and discussions about everything in the world. But, of course, scientific research took priority. N.N. Nekhoroshev was thinking about a very difficult problem. His work did not go easily. But it was the research work initiated by Nekhoroshev in his student years that later brought him worldwide recognition.

After his graduation from the university in 1969, from October of that year to March 1972, N. N. Nekhoroshev pursued postgraduate studies at the Division of Differential Equations under the supervision of V. I. Arnold. In April 1972, he became a lecturer at the same division, and in September 1973 he received his PhD degree. From 1972, N. N. Nekhoroshev, Yu. S. Ilyashenko and E. M. Landis conducted seminars on ordinary differential equations. In the late 1970s, E. M. Landis became less involved with the seminars, while the seminars of Nekhoroshev and Ilyashenko separated. After 2001, N. N. Nekhoroshev worked at the Division of Mathematical Analysis.

With the student virgin land brigade in Kazakhstan in 1965. Nikolaí is the second right, middle row.

One of the first scientific papers of N. N. Nekhoroshev was written in 1971 on the basis of his diploma thesis under the supervision of V. I. Arnold at the Moscow State University. It concerned action-angle variables and their generalizations and turned out to be pioneering in the theory of the so-called noncommutatively integrable systems, i.e., Hamiltonian systems with a complete noncommutative algebra of first integrals. Such a situation occurs, for example, when a system is invariant under the action of a nonabelian Lie group. N. N. Nekhoroshev formulated conditions under which the joint level sets of first integrals are tori with quasi-periodic dynamics and proved the existence of generalized action-angle variables. In the case of the noncommutative algebra of first integrals, in contrast to the standard Liouville integrability, the dimension of these tori is strictly less than one half that of the phase space. Sometimes such systems are also called superintegrable. The ways in which they can be built were subsequently described and numerous examples were given. In each example, the results obtained by Nekhoroshev serve as the principal tool for the description of dynamics.

The PhD thesis of N. N. Nekhoroshev gives the proof of a remarkable and very difficult result on the exponential slowness of the evolution of the action variables in weakly perturbed integrable Hamiltonian dynamical systems. As a hypothesis, such estimates were suggested already by J. E. Littlewood (and possibly also by G. D. Birkhoff). Today these estimates remain the strongest achievement in the multidimensional Kolmogorov – Arnold – Moser (KAM) theory where Kolmogorov tori do not divide the phase space and where phase trajectories can drift slowly in the gaps between these tori. (In astronomic terms, such a drift can correspond to, for example, the Moon falling on the Earth, or to the disintegration of the solar system; KAM theory states that such events are unlikely, while Nekhoroshev's theory assures that even if they are going to happen, they will do it extremely slowly, even on a cosmological time scale.) The exponentially small rate of accumulation of perturbations which was proven by Nekhoroshev is the only reason for the long existence of planets, asteroids, and comets in the neighborhood of the so-called chaotic region. In essence, the time in which the parameters of the principal perturbed motion change substantially is estimated to be larger than an exponent of some power of the quantity which is inversely proportional to the perturbation. Nekhoroshev's theory is based on a remarkable combination of ideas from the theory of Diophantine approximations on submanifolds of Euclidean space and on estimates for Fourier series sums. Nekhoroshev distinguished an important class of unperturbed Hamiltonian systems, which he called steep, and this was instrumental to his success. Every analytic submanifold of Euclidean space which does not belong to any hyperplane turns out to be automatically steep.

Nekhoroshev's estimates are given in terms of rational numbers which he called steepness indices and which characterize the curvature of the manifold.

This work of N. N. Nekhoroshev was honored by the Moscow Mathematical Society award in 1974 and the A. N. Kolmogorov prize of the Russian Academy of Sciences (RAS) in 1997. Nominating N. N. Nekhoroshev for election to the RAS as a Corresponding Member in 1997, Academician V. I. Arnold wrote: "The obvious drawback of N. N. Nekhoroshev's candidacy is his lacking a higher degree, that of DSc. I explain this by the extremely high standards that he demands of himself. And indeed, the level of his PhD thesis surpasses that of the majority of DSc theses." The theory developed by Nekhoroshev rapidly became famous. In 1974, N. N. Nekhoroshev was an invited speaker at the International Congress of Mathematicians in Vancouver, Canada. His works are regarded as a cornerstone in the theory of Hamiltonian systems and have been included in textbooks and monographs. They have been further developed by many mathematicians, specialists in mechanics, and physicists. N. N. Nekhoroshev contributed also to extending many results of his theory to partial differential Hamiltonian equations. This is a similarly remarkable achievement.

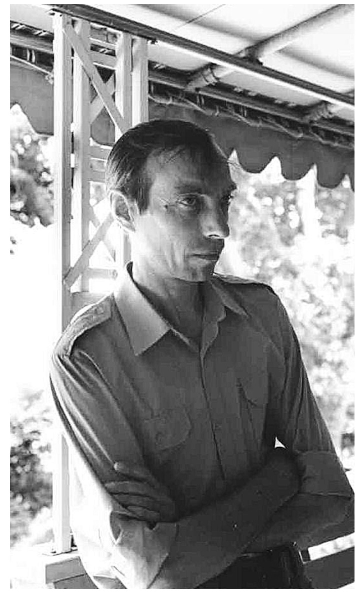
In the following years, N. N. Nekhoroshev continued to work on the theory of Hamiltonian systems. We formulate here one of his results which has turned out to be of great use in soliton studies. Let a Hamiltonian system have a set of commuting first integrals whose number is less than the number of degrees of freedom. Let the phase space contain a smooth compact connected manifold whose dimension equals the number of integrals and which is invariant under all Hamiltonian vector fields generated by these integrals. Then, under certain nondegeneracy conditions, this manifold is a torus which belongs to a family of invariant tori parameterized by the constant values of the integrals. In the phase space, all these tori together form a symplectic manifold, and the restriction of the system to this manifold becomes a standard completely integrable system.

N. N. Nekhoroshev attended many international conferences and worked as an invited professor or researcher at a number of universities abroad. He visited Canada, Cuba, Poland, the United States, Germany, UK, and during his last years France and Italy, where he was professor of the Milano University for three years.

One of the important distinctive qualities of N. N. Nekhoroshev was his interest and willingness to work on the development of concrete problems in applied mechanics and physics. In 2001, during his first encounter with B. I. Zhilinskií at the conference held at the Isaac Newton Institute in Cambridge, UK, he became interested in the mathematical aspects of qualitative models of concrete atomic and molecular quantum systems, such as the hydrogen atom in external homogeneous fields or the system of several coupled angular momenta. His attention was directed primarily to the application of the methods in the theory of integrable Hamiltonian dynamical systems to systems where action-angle variables could not be defined globally.

The problem of the existence of global action-angle variables was analyzed in the first scientific work by N. N. Nekhoroshev which we mentioned above. He had realized already at that time that monodromy was the most basic obstacle to the existence of such variables. Monodromy remained, however, an abstract mathematical concept without possible applications in physical examples which were not given. It was later, thanks to the independent work by J. J. Duistermaat in 1980 and its subsequent development by R. H. Cushman, that physicists became interested in Hamiltonian monodromy and its quantum analogs. Many fundamental systems turned out to be essentially related to monodromy. It should be noted that Duistermaat himself (1942–2010) understood well the contribution by Nekhoroshev, cited Nekhoroshev in his 1980 paper, and took great interest in his late works on monodromy.

During his visit to the French university of Littoral in Dunkirk (Université du Littoral — Côte d'Opale, Dunkerque), where he worked as invited professor at the end of 2001 and at the beginning of 2002, N. N. Nekhoroshev became interested in the formal heuristic generalizations of quantum monodromy which were suggested by his physicist colleagues. Such generalizations follow from the study of joint eigenspectra of two quantum operators, energy and momentum, which correspond to the first integrals of the underlying classical system. In the space of all possible energy-momentum values, i.e., in the base of the integrable fibration, the spectrum forms a two-lattice. Monodromy manifests itself as a (point) *defect* of this lattice centered at the image of the strongly singular point. To uncover monodromy, one considers a closed path in the regular domain of the lattice which goes around the image of this point (the nontrivial cycle of the fundamental group of the base of the fibration). Subsequently, it became customary to demonstrate the defects using a family of elementary (minimal) cells which deform continuously along such a path. Note that the choice of the original cell corresponds unambiguously to the choice of the basis in the first homology group of regular fibers. Monodromy relates the original and the final cells.



In early 2002, after considering possible lattices of systems with resonance 1:2, the idea of a "fractional" defect was put forward. To find out whether and how this can occur, an appropriate dynamical system, the 1:(−2) resonance oscillator with a high-order compactifying term, was suggested. At first, N. N. Nekhoroshev himself categorically objected to the idea that monodromy could be "fractional". (And indeed, we deal with automorphisms of the first homology group H_1 of the torus which are given by integer matrices!) A concrete calculation for the quantum analog of the 1:(−2) system helped him to understand what was happening and to interpret it in terms of the subgroup H_1/Z_2. Subsequently, N. N. Nekhoroshev came to formulate his idea of the proof of usual and fractional monodromy using concrete examples. His approach provides a universal description which relies on the geometric properties of the fibration. The dynamics is used in his proof only to orient the cycles in the homology group (to define the so-called "sign" of monodromy). Essentially, N. N. Nekhoroshev gave a geometric proof of Hamiltonian monodromy which can be related directly to the monodromy concepts in other domains of mathematics. Note that while "usual" Hamiltonian monodromy in the simplest nondegenerate case is locally analogous to the Picard−Lefschetz monodromy of the A_1 singularity, there is no analog of fractional monodromy in singularity theory. Thus, analyzing concrete dynamical systems, N. N. Nekhoroshev generalized the very concept of monodromy. His definition allows for using complete subgroups of the first homology group of regular fibers.

N. N. Nekhoroshev understood well the necessity of solid complete proofs, but above all he strove for conceptual clarity and universality. He used to say: "Let us make sure that our formulation is right! As for the (elegant) proofs—others will follow and come up with them". And they did, fortunately. Afterwards, it took a number of years and a series of joint meetings and discussions in order to complete the proofs for the concrete examples. Very sharp and attentive to details, Nekhoroshev had at the same time a far-reaching vision of the complete work and its further development, and correlated his results to other domains of mathemathics, which he understood and felt profoundly. Aiming at ultimate clarity, he could iterate painstakingly over and over the same paragraph, lemma, or definition. Sometimes, the result of the whole day spent together amounted to half a page, and sometimes it was rejected the next day.

N. N. Nekhoroshev became firmly convinced of the originality of the concept of fractional (generalized) monodromy and began to work actively on its rigorous mathematical foundations. It was the subject of his large last manuscript, which sadly was not published during his life-time. This year, his work has appeared unabridged in Russian in the journal *Russ. J. Nonlin. Dyn.* [vol. 12(3), pp. 413–541 (2016)].

Responding to a referee in early April 2008, Nekhoroshev characterized his work: "I believe that the paper will be useful not only because of its main result, but also because it illustrates a number of rather powerful and universal tools to compute monodromy using a special kind of global section which is similar to the Poincaré section for the trajectories of vector fields." Meanwhile, the idea of fractional monodromy has gained recognition and, as N. N. Nekhoroshev anticipated, new proofs and interpretations have arisen. Generalized monodromy was one of N. N. Nekhoroshev's principal scientific interests during the last years of his life. He is a key figure in the development of the mathematical theory of this phenomenon. The untimely death of N. N. Nekhoroshev on October 18, 2008, has left many possible avenues of further generalizations and applications without continuation. He outlined these possibilities and discussed them with his colleagues. Full of new ideas, he intended to develop them during his new visit to the university of Littoral in May 2008. One week before leaving, a rapid and unexpected deterioration of his health made him cancel his trip. The far-reaching plans remained unaccomplished.

We thank Galina Dyominova, Irina Vasil'eva—the widow of Nikolaí Nekhoroshev, Andrei Kukushkin, Vladimir Piterbarg, and Vera Yashchenko for providing documents and photographs, and for clarifications and commentaries. We have also used the obituary in [Uspekhi Mat. Nauk **64** 3/387 174–178 (2009)]. We are grateful to John B. Delos and Oliver O'Reilly for editing the translation.

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