Quantum monodromy and pattern formation

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Abstract. Hamiltonian monodromy is known to be the first obstruction to the existence of global action coordinates in integrable systems. Its manifestation in quantum systems can be seen as characteristic defects of the regular lattice formed by the joint eigenvalues of mutually commuting quantum operators. Relation between topology of singular fibers of classical integrable fibrations and patterns formed by joint spectrum of corresponding quantum systems is discussed. The notion of the sign of "elementary monodromy defect" is introduced on the basis of "cut and glue" construction of the lattice defects. Special attention is paid to non-elementary defects which generically appear in phyllotaxis patterns and can be associated with plant morphology.

1. Introduction

Hamiltonian monodromy is one of important qualitative features of classical integrable models, characterizing the non-triviality of the fibration of the phase space of classical integrable Hamiltonian dynamical system into common levels of integrals of motion being in involution. Taking into account the fact that Hamiltonian monodromy is a property associated with a relatively rare class of completely integrable dynamical systems the natural immediate question is: Why such particular differential geometry notion can be of interest for applications to real systems in physics, chemistry, or biology, which should generically be described by obviously non-integrable models?

First of all one should remind that integrable approximations become often very accurate in certain regions of the phase space, in particular near the generic equilibrium point, where the motion is regular and toric fibration is typical. According to KAM theory, small deformation of Hamiltonian system does not destroy seriously the regular toric structure [2]. The most of the tori survive small non-integrable perturbation. Similar analysis has been realized for Hamiltonian dynamical systems with monodromy. It has been shown that the monodromy survives under small non-integrable perturbation [6]. The origin of this phenomenon is due to the fact that monodromy is a topological phenomenon and consequently it is quite robust with respect to perturbations. From the point of view of applications this means that the monodromy phenomenon can be observed in real systems which are not obliged to be completely integrable. Even more, as soon as we know what are the fingerprints of monodromy, we can start to analyze some new real physical, chemical or biological systems from the point of view of manifestation of monodromy and then to choose an adequate dynamical model of the phenomenon which should be able to reproduce some important features related to the presence of monodromy.

That is why we start with the manifestations of monodromy in characteristic patterns associated with Hamiltonian dynamical systems.

2. Classical and quantum monodromy

For classical Hamiltonian integrable dynamical systems the presence of monodromy is related to the presence of singularities in the toric fibration associated with the common levels of integrals of motion being in involution [8]. The most direct manifestation of monodromy is the absence of global action-angle variables, or the non-triviality of the fiber bundle structure over the closed contour encircling the singularity on the image of the energy-momentum map [27, 15]. New dynamical manifestation of monodromy related to temporary evolution of a number of individual particles with different initial conditions for a problem with focus-focus singularity was suggested recently [12] but we will treat here "static" manifestation, mainly related to characteristic pattern formation in the joint spectrum of quantum problem corresponding to initial integrable classical one.

To simplify the discussion let us restrict mainly the analysis to a completely integrable Hamiltonian system with two degrees of freedom. Two integrals of motion for such system enable one to construct a momentum map (often named as "energy-momentum map" because one of the integrals is typically an energy) which establishes the correspondence between common levels of these two integrals of motion and the values of these integrals [21, 5]. The momentum map acts from four dimensional phase space of integrable Hamiltonian system to two dimensional space of values of integrals of motion. This map has regular and critical points in the initial phase space of the classical Hamiltonian problem and regular and critical values in the space of values of integrals of motion. Inverse images of regular values are regular tori [2], while inverse images of critical values are various topologically different objects. Consequently, we can say that the momentum map defines a fiber space with the base being the space of allowed values of integrals of motion and the fibers being the inverse images of the map. If now we take the contour in the base space which belong to a simply connected region of a space of regular values of momentum map, the fibration reconstructed over this contour will be trivial and the contour itself is contractible to a point. It is also possible that the contour passes only through regular values of the momentum map but it surrounds critical values. In such a case the fibration over this contour is topologically nontrivial and can be characterized by a monodromy. Monodromy appears as the modification of the basis of the first homology group of the regular fiber after continuation of the basic cycles along a closed contour going through regular values of the momentum map.

The simplest generic isolated critical value of the momentum map is due to presence of a critical point known as focus-focus point. The corresponding singular fiber is a singly pinched torus, i.e. regular torus with one non-trivial (non homotopic to a point) circle shrunk to a point. This singular fiber can be equivalently described as a sphere with one transversal self-intersection point with positive signature.‡ Pinched torus associated with the focus-focus singularity of Hamiltonian integrable dynamical system has by its construction critical point with transversal positive self-intersection. At the same time the matrix representation of the monodromy, i.e. the auto-morphism of the first homology group of a regular fiber, induced by a closed contour in the base space of the toric fibration, depends on the choice of the basis of the first homology group of a regular torus. The basis is defined up to similarity transformation within $SL(2,Z)$ group. This means that the monodromy matrix is defined as a class of conjugated elements of $SL(2,Z)$ group.

The simplest isolated singularity (focus-focus) associated with a singly pinched torus in an appropriate basis leads to monodromy matrix of the form

$$M_- = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{1}$$

The choice of the sign of the non-diagonal element in the monodromy matrix depends on the definition of the orientation of the contour, but it is important that the matrix (1) and the matrix

$$M_+ = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{2}$$

belong to different conjugacy classes of the $SL(2,Z)$ group. It should be noted that for $3D$-integrable systems the monodromy matrices $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ belong to the same class of conjugated elements of $SL(3,Z)$ [41].

As soon as the correspondence between elementary focus-focus singularities and the associated monodromy matrices is found, some natural questions arise. In particular: What monodromy matrices can be realized by multiple or non-elementary singularities of toric fibrations? Does monodromy matrix characterizes the topological type of singularity? We remind here that singularity is considered as a non-elementary one if under a small deformation preserving integrable Hamiltonian structure the singularity can be decomposed into several focus-focus singularities.

Formal answer to these questions can be easily given [29, 10, 11, 42]. Any matrix from $SL(2,Z)$ can be realized as a monodromy matrix of a sufficient number of elementary focus-focus singularities with different vanishing cycles. In such a case the cumulative monodromy matrix associated to a circle surrounding several singularities is a product of a number of matrices corresponding to elementary singularities. One needs only to use the same basis for all singularities. Thus the matrix corresponding to contour surrounding several elementary singularities (but eventually having different vanishing cycles) has the form

$$X = \left(A_1 M_- A_1^{-1}\right)\left(A_2 M_- A_2^{-1}\right)\dots\left(A_k M_- A_k^{-1}\right),$$

where A_i, $i = 1, \dots, k$ are $SL(2,Z)$ matrices transforming elementary M_- matrices to the same basis.

‡ In order to define whether the self-intersection is positive or negative, we need to start by defining a two-dimensional reference frame on a regular point of a fiber and to construct the 4D-reference frame at the self-intersection point by moving initial 2D-frame to critical point in two different non-equivalent ways. If the so defined 4D-frame corresponds to positive volume, the self-intersection is positive. If the volume of the 4D-space calculated with this 4D-frame is negative, the self-intersection is negative.

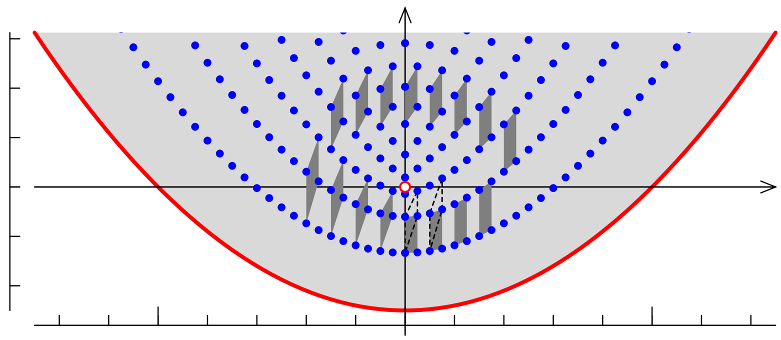

Figure 1. Image of the classical energy momentum map together with quantum joint spectrum of commuting operators for a particle in an axially symmetric (mexican hat like) potential.

In particular even the identity matrix can be obtained as a monodromy matrix associated to a closed contour surrounding $k = 12$ specially oriented elementary focus-focus singularities.

The possibility to get the identity monodromy matrix for non-contractible contour surrounding multiple singularities gives immediately negative answer to second question. Namely, the monodromy matrix does not characterize the topological type of singularity. One needs to add some additional characteristics to distinguish at least between absence of singularities within the contour and the presence of 12 singularities resulting in identity monodromy matrix.

In order to understand better the non-unique correspondence between monodromy matrices and singularities of toric fibration it is quite useful to compare classical picture with corresponding quantum representation [31]. Simple quantization rules for integrable problem enable one to construct a regular lattice of common eigenvalues of two commuting quantum operators in a simply connected region of regular classic toric fibration. This locally regular lattice of common eigenvalues (joint spectrum of mutually commuting observables) can be characterized by an elementary cell. Vertices of an elementary cell correspond to integer values of local action variables which exist for integrable models. Representing joint quantum spectrum on the image of the classical momentum map allows one to associate regions of regular values of momentum map with regions of joint spectrum where the lattice of common quantum eigenvalues can be locally represented as a regular lattice without defects. This is possible in simply connected regions of the image of momentum map which do not include critical values. Transporting quantum elementary cell along a closed contour surrounding some critical value of the classical momentum map allows to find quantum monodromy [31]. Matrix representation of quantum monodromy indicates how the elementary cell transforms after a round trip encircling the singularity. At each step the elementary cell is moved by changing quantum numbers of local action variables by one for all vertices of elementary cell. Matrix giving the quantum monodromy is inverse transpose matrix to classical monodromy matrix giving the transformation of the basic cycles of the first homology group for corresponding classical problem [39, 17, 18].

Thus the quantum integrable problem with monodromy is characterized by a specific pattern of common eigenvalues which can be described as a locally regular

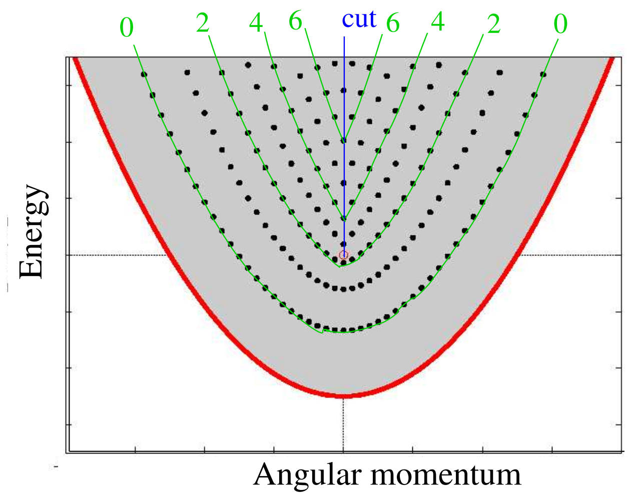

Figure 2. Eigencut for $1:(-1)$ resonant oscillator system represented on the energy-momentum diagram.

pattern with defects [41, 42, 13, 32, 18]. The study of defects of regular lattices becomes, consequently, tightly related to study of singularities of toric fibrations.

3. Lattice defect representation of monodromy

Lattice defects and, in particular, their classification and representation are rather popular subjects in solid state and, especially, in soft matter physics [26]. The common way to represent crystal defect is through the "cut and glue" procedure which normally starts with regular Z^N lattice and after making special cuts and removing or adding parts of regular lattice glues together the so obtained boundaries in order to get lattice with defects.

In the case of lattices formed by joint spectrum of mutually commuting operators with non-trivial monodromy, the locally defined classical action variables enable one to introduce local quantum numbers (n_1, n_2), giving locally two directions of constant action $(n_1 = const, n_2)$, $(n_1, n_2 = const)$. These directions are not globally defined in a unique way in the non-simply connected neighborhood of a singular (defect) point. To represent the lattice with defect we are obliged to make a cut with one end at the singularity and to arrange some rules to cross the cut and to continue the locally well defined lines of constant actions across the cut.

There are two different ways to represent defects of regular lattices formed by joint spectrum of commuting quantum operators in the case of defects associated to isolated focus-focus singularities of corresponding classical systems. The first one is based on the cut made along the so called eigenray (see discussion by Symington [36]). The eigenray is uniquely defined (up to the choice of the direction of the ray starting at the singularity). The values of the actions (quantum numbers) are the same on the two boundaries of the cut within the same local chart if the cut is along the eigenray.

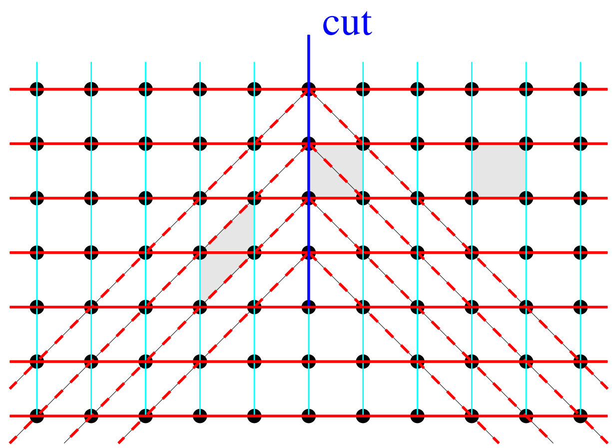

Figure 3. Eigencut for $1:(-1)$ resonant oscillator system represented in action variables. Solid lines should follow dashed lines after crossing the cut.

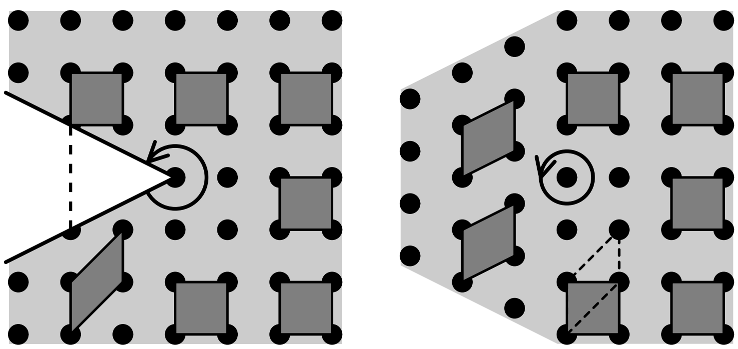

Figure 4. Construction of the elementary monodromy defect by removing solid wedge from the regular lattice.

In contrast, the direction of the "constant action" lines changes when crossing the cut and this direction does not coincide with original directions induced on another boundary by initial local chart. Although the unambiguity of the eigenray has certain advantage from the point of view of defect representation, the resulting geometrical representation (see figures 2, 3) of the constant local action lines shows abrupt change of the slope when crossing the cut. Such representation was used in applications [7] but it makes less clear the construction of the evolution of the elementary cell after crossing the cut.

Another possibility is to keep the directions of "constant action" lines fixed at the two boundaries of the cut at the price of getting different values of actions at two boundaries of the cut in respective points which should be identified after gluing. In such a case the lattice of common eigenvalues in the neighborhood of a singularity is represented in the local chart as a part of regular Z^2 lattice with a particular solid wedge removed and the points on the two boundaries of the removed wedge being identified (see figure 4). Thus the values of actions on two boundaries of the cut are different in the initial local chart with the removed wedge. At the same time the lines of constant actions follow the same direction after crossing the cut. This construction of the monodromy defect was suggested in [31, 9, 42, 28] and used, consequently, in relatively complicated cases [19, 41]. Certain inconvenience of such construction of the defect is an ambiguous choice of the cut and of the removed wedge. At the same time it is important to note that only the geometrical form of the removed wedge is ambiguous. The number of quantum states removed for each subspace with fixed value of another integral of motion is strictly defined. The number of removed states is a linear function of the value of second integral of motion. This statement is in fact the implication of the Duistermaat-Heckman theorem for 2D-degree of freedom integrable classical Hamiltonian systems [16, 21].

4. Duistermaat-Heckman theorem, convexity of energy-momentum map and the sign of monodromy

Restricting again to the case of two degree of freedom integrable classical systems, we can reformulate the statement of the Duistermaat-Heckman theorem as a piece-wise linear behavior of the reduced space volume as a function of the value of the integral of motion. The discontinuity of the first derivative of the reduced volume occurs at those values of integral of motion which correspond to critical values of the energy momentum map. The convexity of the momentum map [21, 40] implies certain restrictions on the type of defects associated with isolated critical values related to focus-focus singularities. Namely, the geometrical representation of "quantum monodromy" was introduced in previous section as construction of a monodromy defect from regular lattice by removing solid wedge and identifying boundaries of the cut. At the same time an alternative construction of somewhat similar defect is possible. It consists in making a cut of the regular lattice and inserting a solid wedge, instead of removing it. From the point of view of matrix representation of corresponding monodromy, the difference between these two formally constructed defects is in the sign of the off-diagonal element of the monodromy matrix. This is the origin of the initial question about possible existence of monodromy with positive and negative sign [10, 11, 42] in integrable Hamiltonian systems, or in more general dynamical models.

In order to formulate this initially not very precise question about the sign of monodromy in a more accurate way, we need first to remind that the correspondence between monodromy matrix and the topological type of the singular fiber is not one to one. This follows immediately from possibility to realize trivial (identity) monodromy matrix with non-contractible closed contour surrounding 12 specially chosen elementary monodromy defects. This means that just comparison of initial and final bases of the homology group of regular fiber is not sufficient to characterize the presence of singular fibers inside the contour. From another point of view, taking in mind the possible construction of defects through removing or adding solid wedge to regular lattice, we can verify immediately that two defects, one obtained by removing and other by adding the same solid wedge, lead to trivial monodromy matrix for closed path surrounding both defects.

To find the difference between positive and negative elementary defects, we need to count the number of states for the reduced space and to plot this function as a function of value of the integral of motion. This function is piece-wise linear and near each singular point there are two possibilities. The function can be convex or concave. We associate with convex function “normal“ defect and with concave function - “inverse“ defect. It is important that multiple “normal“ defects can result in identity monodromy (again 12 elementary specially oriented defects are necessary). In the same way 12 elementary “inverse” defects can also result in identity monodromy. Both examples with identity monodromy are associated with nontrivial fibrations which moreover are non-equivalent between them. For integrable Hamiltonian dynamical systems all elementary monodromy defects should be of the same “normal” type resulting in convex image of the momentum map.

If we suppose that for more general kind of dynamical systems both “normal” and “inverse” defects could coexist, by deformation it could be possible to fuse two defects, one “normal” and one “inverse”. In such a case “normal” and “inverse” defects should annihilate. If such construction is possible remains an open question.

5. Monodromy of *sol*-flower and cat map

An interesting example of an integrable problem with monodromy was recently studied by Bolsinov et al [4]. Namely, the quantum version of the geodesic flow on Sol-manifolds is studied. Authors analyze the main class of Sol-manifolds which are T^2 torus bundles over a circle S^1 with hyperbolic gluing maps with positive eigenvalues. The quantum monodromy is represented on a two-dimensional lattice which can be considered as placed on a cone. The monodromy arises when the contour C surrounds the vertex of the cone and the basis of lattice undergoes transformation A after parallel transform along the closed contour C. The third direction for the Sol-manifold corresponds simply to a trivial extension of the 2D-lattice.

A particular example treated in [4] is related to the cat-map

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{3}$$

The corresponding quantum problem leads to a joint spectrum of two commuting operators which has the form of a regular lattice in any simply connected regular region which is situated away from the origin. At the same time near the origin the actions are not defined globally and continuation of the lines of constant values of local actions along a contour surrounding the origin clearly shows the presence of monodromy. An equivalent way to see the monodromy is to follow the evolution of the elementary cell of the locally defined lattice along the same closed contour surrounding the origin. System of spires corresponding to constant values of local action resembles near the origin the behavior of spires of the sunflower and this similarity was probably at the origin of the name Sol-flower used by authors of [4]. Unfortunately, the resemblance to spiral phyllotaxis observed for many plants (sunflower, pine-apples, pine cones, cabbage, etc is only local (see next section). Comparison of the initial and the final form of the elementary cell after going around the origin enables one to find geometrically the matrix of the corresponding quantum monodromy and to verify that going around the same contour in the reverse way leads to inverse monodromy matrix. Naturally, the form of the matrix depends on the choice of the basis and the quantum monodromy is defined as a class of conjugated matrices with respect to $SL(2, Z)$ group. The choice of the basis made on figure 5 corresponds to standard form A of the cat map transformation given in Eq. 3.

An interesting question now is: Can we replace the effect of a single singularity at the origin for a geodesic flow on the Sol-manifold with cat map A given by Eq.(3) by

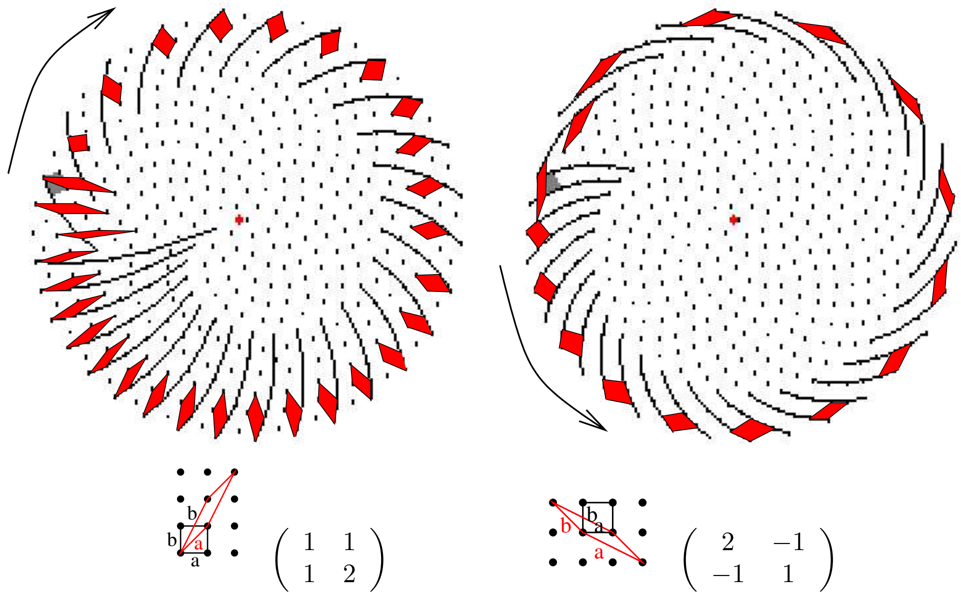

Figure 5. Quantum monodromy for the geodesic flow on the *Sol*-manifold. Evolution of an elementary cell along a closed path going around central singularity in clockwise and counterclockwise directions are shown. Initial and final cells are compared in the bottom of the figure on local part of the lattice which is deformed into regular square lattice.

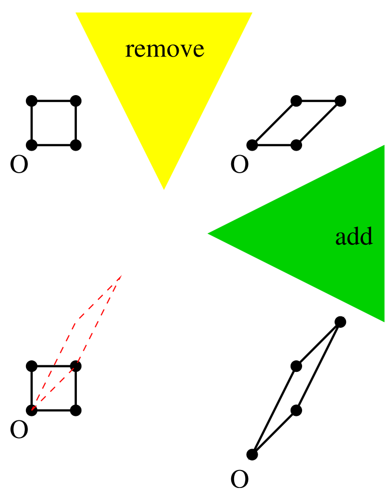

Figure 6. Representation of the cat map monodromy with the aid of one positive and one negative elementary monodromy defect. Initial and final form of the cell coincides with that shown in figure 5, left, but now, during the evolution, there is no 2π rotation of the elementary cell.

a multiple elementary monodromy defects? And what could be the simplest solution?

We can look for a solution by using properties of the modular group $SL(2, Z)$

[29, 11], or by using the geometrical representation of defects through "cut and glue" procedure [42, 41]. Formal algebraic analysis allows to represent "cat map" monodromy in many different ways as a cumulative effect of 12 elementary focus-focus singularities. If we allow to use multiple focus-focus singularities, i.e. singularities having the matrix form $U \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} U^{-1}$, with $U \in SL(2, Z)$ being the matrix describing the modification of the basis of regular Z^2 lattice, and we want to minimize the number of such individual (multiple) singularities, the solution with three multiple focus-focus singularities was suggested in [11] but the authors of that article had no idea how to answer the question about possible minimal number of individual singularities to represent cat map matrix or identity matrix.

Geometrical construction of defects gives a new point of view on the representation of "cat map" monodromy defect or of "identity" defect. First of all we note that the transformation of the initial elementary cell into final cell is associated with the overall 2π rotation of the elementary cell. In order to see better the presence of this overall 2π rotation we can formally construct the same monodromy matrix A of the "cat map" by combining two "elementary monodromy defects", one "normal" and one "inverse". It is also important that these two defects should have different orientation as shown in figure 5. Let us remind that these two defects differ by their sign, and by their geometric representation. One defect corresponds to removing the solid wedge from the lattice, whereas another defect corresponds to adding the same solid wedge to the lattice. Thus the cumulative effect of going around these two defects can be algebraically represented as the "cat map" monodromy A. At the same time, geometrically the evolution of the elementary cell along a closed contour surrounding the pair of "normal" and "inverse" defects differs by 2π rotation form the evolution of the elementary cell going around defect on the lattice corresponding to geodesic flow on Sol-manifold (figure 5).

"Local convexity" arguments forbid the existence of defect corresponding to introducing the solid wedge into lattice. We used such defect just to stress the difference in the global geometry between two defects having the same monodromy matrix representation. Thus, in order to avoid misunderstanding, one needs to add an additional geometrical (topological) invariant to characterize the singularity of dynamical system, the number of 2π rotations of elementary cell during its trip around the singularity.

The simplest dynamical system which can allow the appearance of defects with nonzero number of 2π rotations of the elementary cell should allow the presence of at least 12 elementary monodromy defects. Looking at the list of almost toric symplectic four-manifolds [36, 25] the most interesting candidate is a K3 surface [20]. It appears as a total space of almost toric fibration over two-dimensional S^2 base space with 24 elementary singularities. This means that dynamic system with K3 phase space can be relevant as possible local model of nontrivial non-elementary singularities associated with new nontrivial topological invariant, namely 2π rotation of the elementary cell along a closed path surrounding the singularity. In order to demonstrate the interest in such dynamic models we turn now to completely different examples of biological systems exhibiting so called spiral phyllotaxis phenomenon, which nevertheless seem to be quite related to discussed up to now monodromy and specific pattern formation.

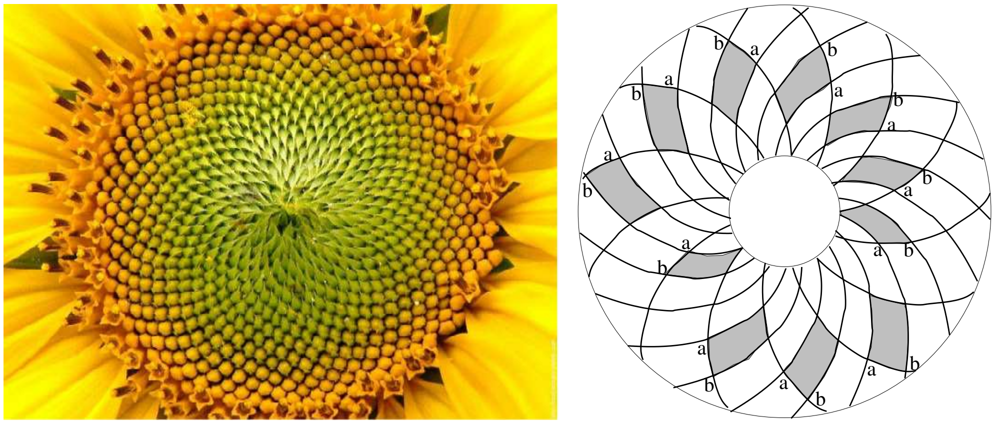

Figure 7. Sunflower with easily eye-guided 34 left and 55 right spires (parastichies) and similar lattice formed by 13 left and 21 right spires together with an elementary cell going around central singularity and exhibiting monodromy showing 2π rotation of the cell. To see better the rotation of the cell, two vertices of all cells are marked by letters a and b.

6. Phyllotaxis and monodromy

Modeling complex biological system, for example the evolution or morfogenesis of plants, is based on a simplification or an idealization which takes into account only some specific particular features and properties of the phenomenon considered. We do not want to enter here into problems of general modeling of morphogenesis (like those suggested by A.M. Turing [38] and R. Thom [37]). Instead we restrict ourselves to study of phyllotaxis (i.e. pattern formation associated with plant development) and more concretely to spiral phyllotaxis which manifests itself through the arrangement of repeated units such as leaves around a stem, scales on a pine cone or on a pineapple, florets in the head of a daisy, and seeds in a sunflower [22]. Spiral phyllotaxis brings always special attention of scientists working in quite different fields. Regular system of helices resembles crystallographic structures and naturally provokes its interpretation in terms of living and growing crystals. Nevertheless, the interest in study of spiral phyllotaxis is mainly due to peculiar appearance of Fibonacci numbers as typical numbers for left and right spirals. The explanation of the appearance of Fibonacci numbers during the self-organization process associated with the plant growth was in fact the main concern in almost all papers dealing with this phenomenon. Many different points of view on this botanical patterns were suggested ranging from purely geometrical or crystallographic till algebraic, dynamical, chemical, genetic, etc. For a review see [1, 22], among more recent relevant papers we can cite [23, 30, 33, 34].

Interdisciplinary character of the phyllotaxis phenomenon can be clearly seen on the example of pattern formation by drops of ferro-fluids in a magnetic field [14] or by flux lattices in superconductors [24].

Here we would like to bring attention of researchers to another aspect of phyllotaxis patterns which was not noted and discussed earlier up to our knowledge, namely its relation to monodromy of lattices with defects and to singularities (Hamiltonian monodromy) of integrable dynamical systems or integrable fibrations in general.

Figure 7 shows real sun-flower with its easily seen spires formed by seeds. There are 55 right and 34 left spires. Locally seeds form an almost regular lattice which can be continued along a closed path surrounding the center of the flower. Similar lattice is reproduced on figure 7, right using another Fibonacci pair of right (21) and left (13) spires. Taking an elementary cell of this lattice and moving it around the center it is easy to see that the elementary cell returns to its original position after making a 2π rotation around itself. This means that the lattice formed by sunflower seeds has a singularity leading to the trivial (identity) monodromy matrix and to non-trivial 2π rotation of the elementary cell with the direction of rotation corresponding to "normal" rotation. This means that the central singularity can be represented as a union of 12 "normal" elementary monodromy defects, arranged in such a way that they produce global identity monodromy matrix.

7. Are there universal non-elementary defects?

We have started this paper by looking at the manifestation of monodromy in very simple dynamic systems with two degrees of freedom. Then we switch to really complex systems and try to look again for monodromy manifestations just by studying the patterns showing singularities. We interpret these singularities as a non-elementary defects and represent them as a cumulative result of several elementary monodromy singularities. The idea behind such analysis is to find an adequate mathematical tools to describe and to characterize complex systems in terms of relatively simple models.

One of the typical ways to understand the behavior of complex systems is to find and to describe universality classes of the organization of complex systems. The scale invariance in particular is known to be appropriate not only for physical phase transitions and critical phenomena but for a number of systems arising in disciplines as diverse as biology, ecology, economics, cardiovascular medical systems, etc. By scale invariance is meant a hierarchical organization that results in power-law behavior over a wide range of values of some control parameter such as species size, heartbeat interval, or firm size, and the exponent of this power law is a number characterizing the system [35].

Looking at simple dynamical systems with monodromy, the natural question arises: What kind of behavior we can expect for complex systems from the point of view of singularities and more specifically of corresponding monodromy. It is quite reasonable to suppose that the number of individual elementary singularities should increase when going to more complex systems, but at the same time it is probable that formation of certain groups of singularities can be favored and the systems possessing these non-elementary groups of singularities can be classified into universality classes. We suppose that the number of 2π rotations of the elementary cell during a closed path around a singularity (group of singularities) can be considered as a universality class of dynamical systems. In this sense all plants exhibiting spiral phyllotaxis belong to the same universality class. Mechanical systems possessing only monodromy with one vanishing cycle (in the case of several elementary singularities all vanishing cycles should be the same) belong to another universality class (zero number of 2π rotations).

Extension to problems with higher number of degrees of freedom, i.e. to three and higher dimensional patterns leads naturally to question about defects of higher codimension. No examples of dynamical systems with "elementary defects" of higher codimension, generalizing the codimension two monodromy, are known to the author, except some formal construction [3].

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