Monodromy and complexity of quantum systems

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1   Introduction

In order to see better the place of the present chapter within the very broad subject of complex dynamical systems we need to start with a short discussion of three key notions: Quantum systems, Complexity, Monodromy; and to explain their interrelation and relevance to the general study of dynamical systems.

By quantum systems we mean here very simple objects formed by a finite number of particles, typically atoms and molecules, which exist in bound states and can be described within the standard non-relativistic quantum mechanics. This means that the system can exist in a number of states, characterized by discrete values of certain physical quantities, or equivalently by a set of quantum numbers. The physical quantities are represented in quantum theory by operators, whose eigenvalues give the possible values of these quantities. The operators do not generally commute. In order to characterize the quantum state we need to use the eigenvalues of a set of mutually commuting operators.

The finite number of degrees of freedom and even the finite numbers of quantum states which we are typically interested in lead to an impression that such dynamical systems have more chance to be treated as "simple dynamical systems" rather than complex. Thus an example which will be mentioned, the hydrogen atom, is one of the simplest real quantum systems with a completely regular set of quantum states forming highly degenerate groups of levels, so called shells, which are present due to specific dynamic symmetry of the problem. Such shells themselves can be considered independently as quantum dynamical systems with two degrees of freedom using effective Hamiltonians. At the same time even a small external perturbation of a hydrogen atom by constant electric and magnetic field leads to a splitting of the degeneracy and to the appearance of a complicated pattern of eigenstates strongly dependent on the values of external parameters characterizing the perturbation.

More generally, the analyzed system of quantum states can be considered as a simple one if all states can be arranged in regular patterns characterized by several good quantum numbers taking consecutive integer values. Such quantum systems correspond to classical Hamiltonian integrable systems whose dynamics is regular and is associated with toric fibrations. To go to the extremely simple

case one can assume that classical action-angle variables are globally defined. For quantum systems this means that the set of quantum states represented through the joint spectrum of quantum operators can be globally arranged in a regular lattice whose nodes are characterized by quantum numbers associated with eigenstates of mutually commuting quantum operators corresponding to classical integrals of motion.

Another natural limiting behavior of dynamic systems is associated with the irregular spectra of quantum systems, or with the irregular or chaotic motion of classical dynamical systems [48]. Such irregular or chaotic dynamics is usually qualified as much more complex than the regular motion. But generically the majority of dynamical systems exhibit both regular and chaotic behavior at different energies and/or at different values of control parameters. The study of the transition from the regular to the chaotic limit should be done for parametric families of dynamic systems. Such a transition typically goes through several consecutive steps from completely regular motion through a partial breaking of integrals of motion [32]. At the same time, even at the level of completely regular motion for completely integrable systems, the complexity of a dynamical system can be increased through a sequence of bifurcations (in both classical and quantum cases [51, 67]). Increasing complexity of a completely integrable classical (or quantum) dynamical systems can be most easily seen by analyzing the image of its energy-momentum map (or bifurcation diagram) [8, 5].

For a completely integrable classical Hamiltonian dynamical system the energy momentum map establishes the correspondence between common levels of energy and other integrals of motion and the values of these integrals which are in involution for a completely integrable classical system. We remind that two physical quantity are in involution in classical mechanics, if their Poisson bracket is zero. In quantum mechanics, the corresponding operators commute. The energy momentum map has regular and critical points in the initial phase space of the classical Hamiltonian problem, and regular and critical values in the space of values of integrals of motion. Inverse images of regular values are regular tori [2], while inverse images of critical values are various topologically different objects. Consequently we can say that the energy momentum map defines a fiber space with the base being the space of allowed values of integrals of motion and the fibers being the inverse images of the map. The set of critical values of the energy-momentum map defines in some sense the complexity of the integrable dynamical system. In order to characterize the complexity we need to describe either the topology of singular (critical) fibers and their organization, or to introduce some special characteristics of a family of regular fibers surrounding the above mentioned singularities. This is exactly the moment when the Hamiltonian monodromy appears as a natural characteristic to describe the singularities of toric fibrations and the complexity of dynamical systems.

For quantum analogs the system of common eigenstates of mutually commuting quantum operators which form the complete set of observables plays the role of toric fibration. Locally the joint spectrum of mutually commuting operators forms a regular lattice in regions corresponding to regular toric fibrations of associated classical systems. At the same time, singularities of classical

toric fibrations correspond to certain defects of the pattern representing the joint spectrum of quantum operators. We can equally try to describe the defect by analyzing the behavior of the locally regular lattice surrounding the defect. This leads us naturally to quantum monodromy which generalizes the classical Hamiltonian monodromy to quantum systems.

After this intuitive introduction giving initial ideas about relations between the complexity and monodromy of classical and quantum systems and singularities (or defects) of almost regular patterns associated with classical and quantum systems we turn to a slightly more detailed description of the Hamiltonian monodromy along with more concrete examples of classical and quantum dynamical systems possessing monodromy. We will even try to generalize the characteristic discrete patterns with monodromy which arise naturally for quantum atomic and molecular systems and to look for generic universal patterns and their defects which appear in completely different scientific domains like botany, but which nevertheless can be considered as a result of the evolution of a complex dynamical system showing simple universal behavior.

The organization of the paper is as follows. Section 2 deals with basic introduction to classical Hamiltonian monodromy. Classical quantum correspondence is used in Section 3 in order to explain associated quantum monodromy. Interpretation of quantum monodromy in terms of defects of lattices of quantum states and associated discussion of elementary and complex defects is the subject of the central for this paper section 4. Possible generalizations of the monodromy concept, naturally arising during the analysis of concrete physical examples, are suggested in section 5. Short section 6 reviews recent progress in monodromy manifestations in time dependent processes. The last section 7 discusses perspectives of possible applications of the monodromy concept to very complex biological phenomena like plant morphogenesis, using as an example one particular but universal phenomenon: spiral phyllotaxis.

2   Hamiltonian monodromy

The notion of "monodromy" is generally used in science, mainly in mathematics, in order to explain how some mathematical objects behave as they "go" around a singularity. This is a very general and unprecise description which can be made more concrete within the fiber space construction [42, 8].

Let $p : E \rightarrow B$ be a locally trivial fiber space with the base B. With each point $b \in B$ we associate a fiber $p^{-1}(b)$ and with each continuous path $\gamma : [0,1] \rightarrow B$ in B with initial point $a = \gamma(0)$ and end-point $b = \gamma(1)$ we can associate a homeomorphism of the fiber $p^{-1}(a)$ onto the fiber $p^{-1}(b)$. In a particular case of $a = b$, the path γ is a loop and we have a transformation of $F = p^{-1}(b)$ into itself. This transformation defined up to a homotopy is called the monodromy transformation. The transformation of fibers induces the transformation on the homology and cohomology spaces of F, which are also called a monodromy transformation.

In the context of classical Hamiltonian integrable dynamical systems the

monodromy transformation appears naturally as a transformation of a regular fiber associated with a closed path in the base space defined as a space of possible values of integrals of motion being mutually in involution [8, 17]. Image of the energy momentum map (known also as bifurcation diagram [5]) enables us to clearly visualize for 2D-integrable Hamiltonian systems the qualitative structure of the associated fiber space. Regular fibers in this case are regular tori [2] and the monodromy transformation can be described as a transformation of a fundamental group (or a homology group) of a fiber associated with closed paths in the base space. In a more classical mechanics way we can speak about local action-angle variables defined on individual tori and their connection (evolution) along a family of tori associated with a loop in a base space. The fact that a loop goes around a singularity and is consequently noncontractible implies the absence of global action-angle variables and the presence of non-trivial monodromy transformation [43, 17].

The simplest situation in the case of completely integrable dynamical systems with 2 degrees of freedom corresponds to the presence of an isolated singular fiber (see Figure 1 a) which can be surrounded by a loop in the base space going only through regular fibers [40, 5, 19, 61, 63]. Such a situation is present, for example, in a number of very simple model mechanical problems like particle motion in an axially symmetric potential of "mexican hat" or "champagne bottle" type : $V(r) = ar^4 - br^2$, with $a, b > 0$ [3, 6], or for spherical pendulum [8, 9, 19]. We can associate with each isolated singular fiber a closed path going around a corresponding isolated critical value in the base space. In its turn this closed path is associated with the transformation of the first homology group of a regular fiber. This transformation can be naturally represented by a matrix with integer coefficients leading to integer monodromy. A typical isolated singular fiber which appears generically for Hamiltonian systems with two degrees of freedom is a pinched torus represented in Figure 2 a. A pinched torus is obtained from a regular torus by shrinking one non-trivial circle to a point. Its geometrical view as an object in the three-dimensional ambient space can give an impression that depending on the choice of vanishing cycle the geometry of the pinched torus can be different. But this is just an artifact which is caused by plotting the pinched torus in 3-D rather than in 4-D. In fact, the choice of coordinates on the torus is ambiguous and this ambiguity is due to the $SL(2, Z)$ symmetry of a two-dimensional lattice. This leads, in particular, to the important fact that the matrix representation of monodromy is defined up to a $SL(n, Z)$ similarity transformation for n-dimensional problem.

It is possible that instead of one isolated singular fiber the image of the energy-momentum map is a whole region with a complicated set of singular fibers surrounded nevertheless by a regular region. Figure 1 c shows an example of an "island" which can be surrounded by a closed loop in the base space going through only regular fibers. The resulting monodromy transformation characterizes the whole region possessing singular fibers and in such a case we can speak about nonlocal integer monodromy [53, 36]. One of the simplest reasons of the appearance of a nonlocal "island" like singularity is the formation of a second component in the image of the energy-momentum map due to the
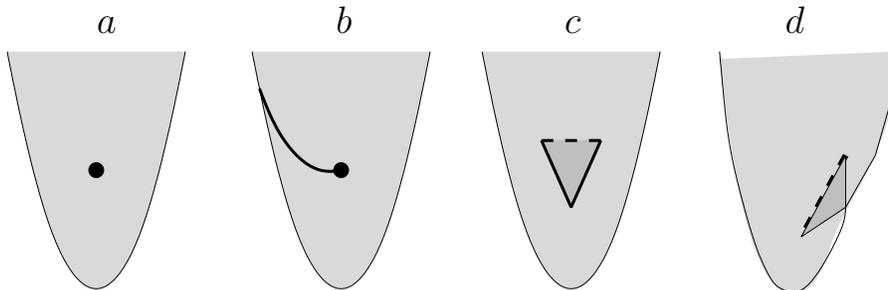
Figure 1: Typical images of the energy momentum map for completely integrable Hamiltonian systems with two degrees of freedom in the case of: a - integer monodromy, b - fractional monodromy, c - nonlocal monodromy, and d - bidromy. Values in the light shaded area lift to single 2-tori; values in the dark shaded area lift to two 2-tori. Dashed lines in sub-figures c, and d correspond to stratum formed by bitori.

transformation of an isolated focus-focus singularity into an "island" through the Hamiltonian Hopf bifurcation [19] which is related to the presence of a family of singular fibers. Each generic member of such a family is named a bitorus. It is represented in Figure 2 c. A singular (cuspidal) torus (see Figure 2 d) is its limiting form corresponding to corners on the bifurcation diagram 1 c where the bitorus line ends.

A less trivial situation arises when the essentially singular fiber is not isolated but appears as a limiting case of a one-dimensional stratum of weakly singular fibers [45, 46, 19]. The presence of weakly singular fibers does not allow one to go around the essential singularity by transporting the basis cycles of the homology group of regular fibers. At the same time it is now possible to study the continuous evolution of cycles for certain subgroups of the homology group. Such construction allows us to generalize the monodromy notion and introduce "fractional monodromy" [45, 46, 21, 29, 44, 60]. An image of the energy-momentum map with a singularity leading to fractional monodromy is shown in Figure 1 b. An example of weakly singular fibers, the so called curled torus, is shown in Figure 2 b.

Qualitative characterization of images of energy-momentum maps for classical integrable systems and especially of their generic possible evolution for families of integrable systems depending on one or several control parameters is important from the point of view of different generalizations. In spite of a very serious restriction due to integrability, the qualitative features, like monodromy, remain valid even for non-integrable systems which can be obtained by small non-integrable perturbations [4]. The situation here is similar to the KAM theorem [2] assuring that the majority of tori survive under a small perturbation leading from regular to chaotic classical motion.
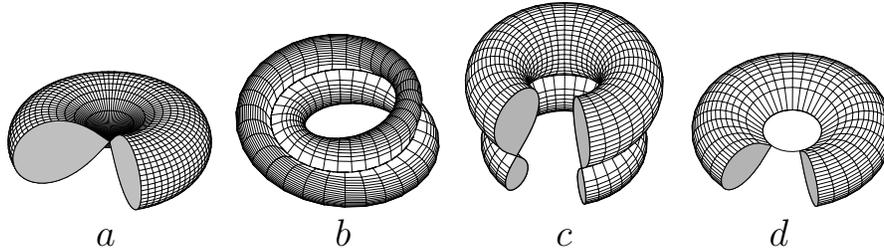
Figure 2: Two dimensional singular fibers in the case of integrable Hamiltonian systems with two degrees of freedom: a - pinched torus, b - curled torus, c - bitorus, and d - singular (cuspidal) torus.

3   Classical quantum correspondence

Another important aspect of classical integrable systems with monodromy is the generalization from classical to quantum mechanics [9, 61, 62, 52, 28]. The quantum analog of a classical integrable dynamical system is a quantum system possessing a mutually commuting set of operators corresponding to physical observables. Classical local action variables correspond in quantum mechanics to quantum numbers which label common eigenfunctions of mutually commuting operators. Representation of joint spectra of mutually commuting quantum operators naturally leads to locally regular lattices due to the simple quantization conditions imposed on local classical action variables.

Formal correspondence between classical integrable Hamiltonian systems and their quantum analogs helps to visualize the quantum monodromy [52]. Existence of local actions for an integrable system means that the joint eigenvalues of mutually commuting quantum operators corresponding to classical integrals of motion for a n-degree of freedom, completely integrable, dynamical system form a pattern which locally can be mapped onto a standard Z^n lattice [66, 65]. If the quantum commuting operators correspond directly to classical actions the lattice formed by the joint spectrum of these operators represents a regular rectangular pattern because of the simple quantization rules for both action variables. A more typical situation corresponds to cases when only part of the classical integrals of motion are actions, while others, like energy, are not actions themselves, but can be locally transformed into actions by nonlinear (and in some sense small) transformations.

Figure 3 gives an example of classical - quantum correspondence for completely integrable problems with two degrees of freedom. The image of the classical energy momentum map is shown in the space of values of integrals of motion: {energy, first action}. Figure 3 a shows the regular part of the image of classical energy-momentum map together with points corresponding to joint eigenvalues of quantum mutually commuting operators. In order to see the natural correspondence of a discrete lattice with regular Z^2 lattice, an elementary cell of the lattice is chosen and displaced through the lattice along a closed path.
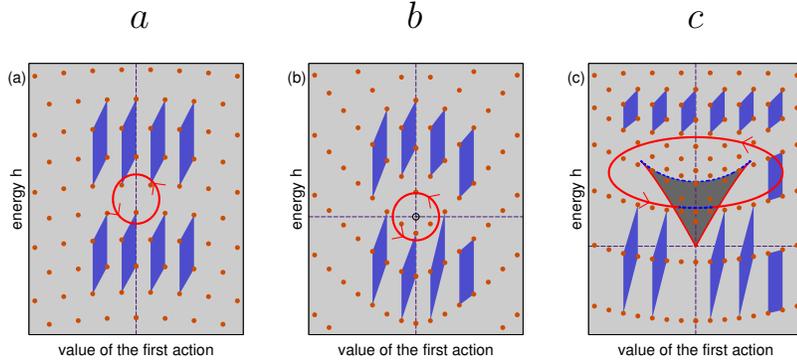
Figure 3: Quantum joint spectra for typical regions of the image of the energy - momentum map for some examples of integrable problems [15].

It is evident that the initial and final cells coincide and all closed paths in this local regular region of the lattice are similar from this point of view. In more formal terms we can say that all closed paths in the regular simply connected region of the image of the energy-momentum map are homotopic to a point, i.e. such closed paths could be shrinked to a point because all intermediate cells remain equivalent.

Figure 3 *b* illustrates another possibility. The region of the classical energy-moment map shown here has an isolated singularity, associated with a pinched torus. In any simply connected local region which does not include the singularity, the behavior of the quantum cell after going around a closed path is similar to that shown in Figure 3 *a*. In contrast, if the closed path goes around the singularity the final form of the elementary cell clearly differs from the initial form, thus indicating the presence of nontrivial monodromy [52]. The transformation between initial and final cells does not depend on the geometry of the closed path supposing that it goes only once around the singularity. The transformation between initial and final cells written in the matrix form gives the matrix representation of quantum monodromy. Naturally, the explicit form of the matrix depends on the choice of the lattice basis which is defined up to an $SL(2, Z)$ similarity transformation.

Figure 3 *c* shows the image of the classical energy - momentum map with a region where two connected components exist in the inverse image. This curvilinear triangular region is represented in the figure by dark hashing. The correspondence between classical and quantum mechanics is more complicated for this example. Outside the region with two classical connected components there is one lattice of quantum joint eigenvalues. Inside the "dark" region with two classical connected components of the inverse image all quantum states can be approximately separated into two distinct leaves (neglecting the tunneling splitting which allows the coupling of quantum states associated to two disconnected classical components). Taking an elementary cell on the big leave
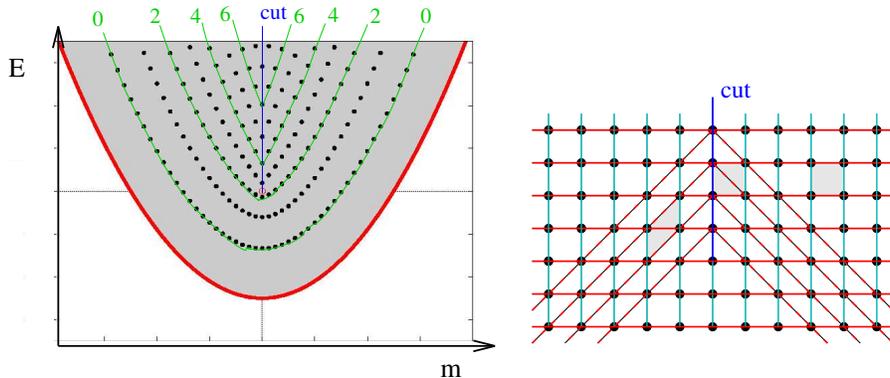
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Figure 4: $1 : (-1)$ resonant oscillator system with a cut along a *eigenray* $R = \{m = 0, E > 0\}$. Left: Representation in energy - angular momentum variables. Right: Representation in action variables.

and going around the "island" we define a quantum nonlocal monodromy which characterizes the second leaf of the lattice and the way how two leaves join together. The vibrational structure of an LiCN molecule can be suggested as an example of a concrete molecular system which is reasonably well described by an integrable approximation which shows the presence of an island and a nonlocal non-trivial monodromy [36]. The second component exists as well for the quite similar example of an HCN molecule [22] but the closed path surrounding the second component cannot be constructed for the HCN model and consequently the HCN example is completely different from the LiCN one.

4   Lattices and defects

The above mentioned examples give an impression that patterns formed by the joint spectrum of mutually commuting quantum operators can be considered either as ideal periodic lattices (ideal crystals) or as periodic lattices with defects [66]. More generally one can suppose that lattice defects of periodic crystals [41, 37] should correspond in some way to singular fibers of classical integrable dynamical systems. In fact, this analogy proves itself to be extremely useful, although even the simplest classical singularity associated with a pinched torus (so-called focus-focus point) leads to a defect of the lattice of quantum eigenstates which has no straightforward analog among typical crystal defects.

4.1   Elementary monodromy defects

In order to see the correspondence with solid state defects we analyze the pattern of quantum eigenstates for a two degree of freedom integrable system possessing, in the classical limit, a focus-focus singularity. A typical classical image

of the energy momentum map together with common eigenstates of mutually commuting operators for corresponding quantum problem are shown in Figure 4. To demonstrate the presence of the defect, we do the cut starting at the singularity of the classical problem. There are a lot of different possibilities to realize such cut. We first do the cut along the $m = 0, E > 0$ ray. Such choice of the direction of a cut is named an *eigenray* by M. Symington [63]. Its specificity can be easily explained by using the Figure 4, where we have labeled eigenstates for each m by consecutive integers starting from zero. These integers play the role of quantum numbers associated with a second action which is defined locally in any simply connected region of regular values of the energy-momentum map. We see that this local action (quantum number) has the same value for each eigenstate located at the cut irrespective of either we approach the cut from the right or from the left. At the same time the first derivative $\frac{\partial E}{\partial m}$ (calculated for a set of states with the same quantum number corresponding to a second action) has a jump at the cut. Due to the presence of such a discontinuity of the first derivative this cut was named "kink" line by M. Child [6]. It should be noted that the presence of a "kink" singularity is a purely artificial fact related to the multivaluedness of the action variables and to our choice of one leaf of the multi-valued function.

In fact, we can continue labeling eigenstates of the joint spectrum by continuing local action variables within $E > 0$ energy region from the $m > 0$ to the $m < 0$ domain. This will naturally give another labeling scheme which can be shown on the bifurcation diagram and which is associated with an alternative construction of a single valued function from an initially multivalued one. If we transform the (E, m) plot to new coordinates which we choose as two local actions, [as it is shown in Figure 4, right] in order to continue the line corresponding to one chosen value of local action across the $(m = 0, E > 0)$ ray we need to change the direction after crossing the ray.

If we do cut along any other direction, the local action itself has a discontinuity at the cut. This situation is shown in Figure 5. Further transformation of the lattice with a cut to local action coordinates leads to a regular lattice with a certain part of this lattice removed and with the boundaries of this removed wedge being identified [65, 66, 46]. Figure 6 illustrates the construction of the defect associated with the simplest focus-focus singularity (single pinched torus) in the regular lattice by making a cut in the direction orthogonal to the "eigenray". Such construction is similar in spirit to the representation of dislocations and disclinations in solid state physics by a cutting and gluing procedure. At the same time the obtained defect differs from well known constructions for dislocations and disclinations. We named the defect shown in Figure 6 an "elementary monodromy defect". The most important features of the suggested construction of an "elementary monodromy defect" is the linear dependence of the number of removed states on the value of the integral of motion. This property is related to the Duistermaat-Heckman theorem applied to the volume of the reduced phase space in classical mechanics [18, 62, 31].
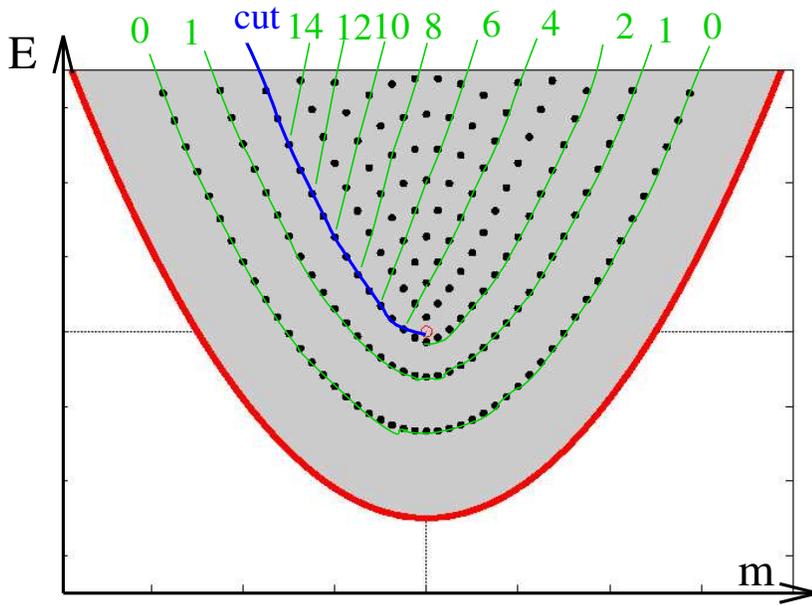
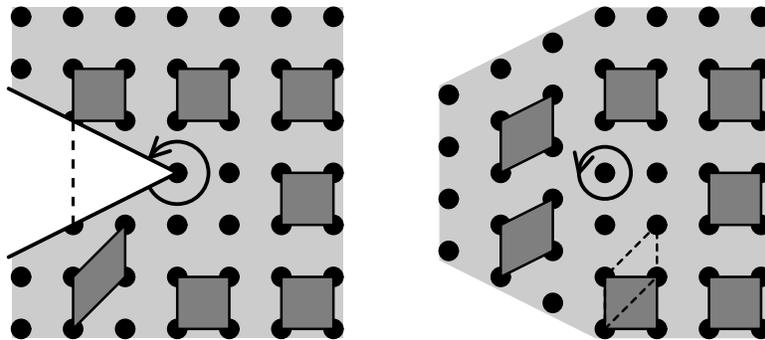Figure 5: Cut leading to a discontinuity of the second local action.

Figure 6: Construction of the 1:(−1) lattice defect starting from the regular Z^2 lattice. Dark grey quadrangles show the evolution of an elementary lattice cell along a closed path around the defect point.

4.2 Fractional monodromy defects

The geometrical construction proposed for the description of a monodromy defect allows immediately several alternative ways of generalization. The first possibility is the construction of less trivial elementary defects, like recently introduced fractional monodromy defects [45, 46]. We illustrate construction of such a defect in Figure 7. The idea of relevance of such a specific defect to patterns formed by joint eigenvalues of mutually commuting operators came from the typical dependencies of the number of states in multiplets or polyads on the quantum number characterizing these polyads or multiplets. This question is again related to the Duistermaat-Heckman approach describing the evolution of the reduced phase spaces with the corresponding integral of motion value.

The important difference between the removed wedge associated with elementary "integer" monodromy (represented in Figure 6) and the removed wedge in the case of a fractional monodromy defect (see Figure 7) is in the number of the removed states considered as a function of the integral of motion. In the integer case this function is linear [or polynomial in the higher dimensional case] [31], while in the fractional case the function is a "quasi-polynomial" [59], i.e. it includes an oscillatory part. In a particular example shown in Figure 7 we have just modulo 2 oscillations. In an equivalent way it is possible to say that we need to consider separately the sub-lattices with even and odd m values, each possessing an integer "elementary monodromy defect". The geometrical consequence is the impossibility for an elementary cell to go unambiguously through the cut. Depending on the position after crossing the cut, the elementary cell takes one of two different geometrical forms shown in Figure 7, left. In contrast, if we use a double cell, the result of the cell transformation after making a closed trip around the essential singularity and crossing the cut only once is independent of the place where the cell crosses the cut. The price for that are the fractional coefficients which appear in the monodromy matrix written for the elementary cell. We can formally use the monodromy matrix for an elementary cell in the regular region even if this elementary cell itself cannot cross the singular cut.

An example of a simple model problem showing the presence of a fractional monodromy and the associated pattern of common eigenstates of two mutually commuting operators is shown in Figure 8 [33].

The corresponding dynamical system is constructed by taking two angular momentum operators $\mathbf{N} = (N_x, N_y, N_z)$, $\mathbf{S} = (S_x, S_y, S_z)$ interacting in a nontrivial nonlinear way between themselves and with an external field. The model Hamiltonian is given as:

$$H_\lambda = \frac{1-\lambda}{|\boldsymbol{S}|} S_z + \lambda \left(\frac{1}{|\boldsymbol{S}||\boldsymbol{N}|} S_z N_z + \frac{1}{2|\boldsymbol{S}||\boldsymbol{N}|^2} \left(N_-^2 S_+ + N_+^2 S_- \right) \right), \quad (1)$$

where standard notation is used for ladder operators $N_\pm = N_x \pm iN_y$, $S_\pm = S_x + iS_y$, and $|\mathbf{N}| = \sqrt{N_x^2 + N_y^2 + N_z^2}$, $|\mathbf{S}| = \sqrt{S_x^2 + S_y^2 + S_z^2}$. This model generalizes the most trivial model of angular momenta coupling leading to the appearance of a focus-focus singularity in classical mechanics and an integer
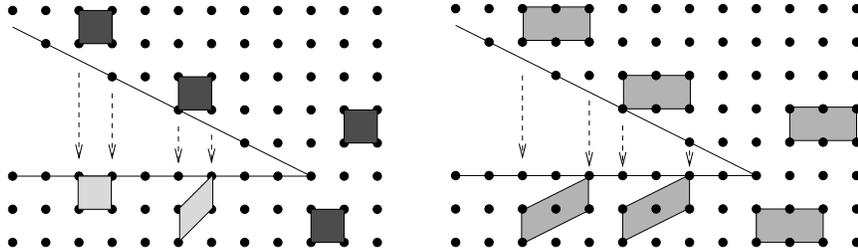
Figure 7: Construction of 1 : 2 rational lattice defect. Left: Elementary cell does not pass unambiguously. Right: Double cell passes unambiguously.
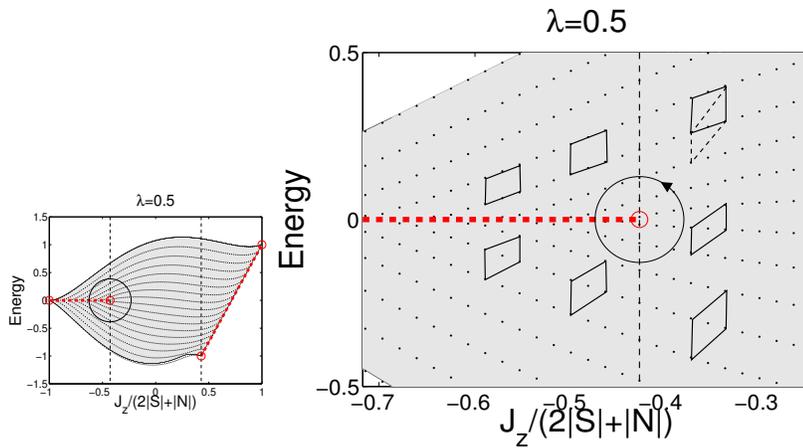

Figure 8: Fractional monodromy for two angular momenta coupling model. Image of the energy-momentum map is given for Hamiltonian (1) with $\lambda = 0.5$.

quantum monodromy in the corresponding quantum problem [47, 52, 26, 27]. It is interesting to note that the simplest quantum problem, namely the hydrogen atom in the presence of external electric and magnetic fields leads to integrable approximations for certain values of field parameters, which show the presence of fractional monodromy [25, 23, 24].

4.3 Monodromy - defect correspondence

Another way of generalizing the possible patterns typical for lattices formed by joint spectrum of mutually commuting operators is to describe more complicated defects which can be present for 2D-lattices from one side and for higher dimensional lattices from another side. We restrict ourselves here with 2D-case only.

First of all it should be reminded that if we want to characterize the de-

fect of a quantum lattice (or the singularity of a classical toric fibration) by its monodromy, we introduce in some sense a certain equivalence relation between different defects (singularities). The most detailed description of a singularity of classical toric fibration and of the associated quantum state lattice can be done by indicating the explicit transformation of the basis of the first homology group of regular fibers. In such a case two defects characterized by the presence of one vanishing cycle become different if the vanishing cycles themselves are different. Naturally, the monodromy matrices associated to these two defects and written in the same basis of cycles (or on the same lattice basis) are different. At the same time these matrices belong to the same class of conjugated matrices within the $SL(2, Z)$ transformation group responsible for the basis transformation of the 2D-lattice. Thus, the monodromy matrices should be considered equivalent if they belong to the same class of conjugated elements of the $SL(2, Z)$ group. To characterize the class of conjugated elements we use first the trace of the matrix. The matrices $M \in SL(2, Z)$ are named parabolic, elliptic or hyperbolic, if their traces equal ± 2 (for parabolic), $\pm 1, 0$ (for elliptic), or $> |2|$ (for hyperbolic). A more detailed classification of parabolic matrices into classes of conjugated elements needs additional information, like a non-diagonal element of some standard "normal form" of the parabolic matrices. Especially interesting is the importance of the sign of the non-diagonal element indicating that, for example, two quite simple monodromy matrices

$$
\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}
$$

belong to different classes of conjugated elements within the $SL(2, Z)$ group. We name these matrices positive and negative elementary monodromy matrices. It is important to note that only one class corresponds to an elementary toric singularity, the pinched torus. Its representation as a defect of a regular lattice corresponds to removing a wedge from the lattice and to identifying the wedge boundaries. It is also important to note that using several elementary monodromy matrices which belong to the same class of conjugated elements of $SL(2, Z)$ but can be reduced to the normal form in a different lattice basis it is possible to construct an arbitrary $SL(2, Z)$ matrix representing a cumulative monodromy effect [12, 11].

At the same time the monodromy (i.e. the class of conjugated elements of $SL(2, Z)$ group) does not define in a unique way the classical singularity and the pattern formed by the joint spectrum of commuting quantum operators [11, 12]. For example, 12 specially constructed elementary singularities (each described by a matrix conjugated to the simplest monodromy matrix and represented by defects associated with removing a wedge from the lattice) can lead to global trivial monodromy. This is a simple consequence of the $SL(2, Z)$ group structure [55, 49]. Non-triviality of a global singularity nevertheless is clearly seen through the transformation of the elementary cell going around all singularities on the lattice of mutual quantum eigenstates. The cell realizes the 2π rotation around itself while going along a closed path surrounding these 12 elementary
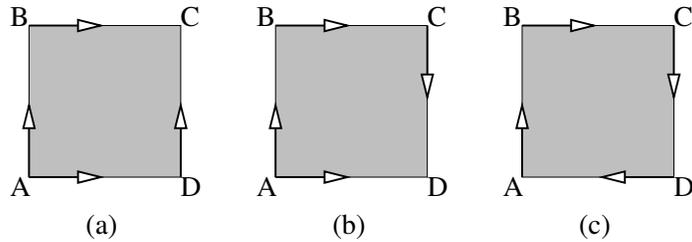
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Figure 9: Construction of two-dimensional surfaces by the identification of the opposite ends of a square, preserving the directions indicated by arrows. (a) - Identification of AB with DC and AD with BC gives torus. (b) - Identification of AB with CD and AD with BC gives Klein bottle. (c) - Identification of AB with CD and BC with DA gives real projective plane, RP_2.

singularities. It is important that the sign of the rotation of the elementary cell is well defined and corresponds to removing wedges from the lattice [65].

As another example, it is possible to realize the negative "elementary monodromy" by using 11 elementary positive monodromy defects. In such a case the rotation of an elementary cell while going around this defect can be again characterized as a "positive" and being almost overall 2π rotation in spite of the fact that a much more simpler construction of the elementary "negative defect" can be formally done by inserting into the lattice a wedge, corresponding to one elementary monodromy defect, instead of removing from the lattice 11 wedges corresponding to elementary defects as in the case of "positive defects".

The notion "elementary monodromy" which we use is due from one side to the simplicity of the matrix representation but from another side due to simplicity of the topology of the singular classical fiber responsible for such monodromy. The geometric monodromy theorem states that the presence of an isolated singly pinched torus leads to the elementary positive monodromy of an associated toric fibration [8, 40]. A singly pinched torus can be alternatively described as a sphere with one transversal positive self-intersection point. In order to understand this statement better it should be reminded that in four-dimensional space the 2-dimensional surfaces generically intersect via isolated points and the simplest model of a pinched point of a torus corresponds to the intersection of two 2-D planes ($x = 0, y = 0$) and ($z = 0, w = 0$) in 4-D $\{x, y, z, w\}$-space. Positivity of the self-intersection point means that a 4D-frame constructed from two 2D-frames transported from a regular point on the surface to the singular point of self-intersection via two non-equivalent paths gives positive volume.

The representation of a negative "elementary monodromy" in terms of 11 elementary positive singularities shows in some sense the complicated nature of the "elementary negative defect". At the same time simple topological arguments suggest, instead of a singly pinched torus as a possible candidate for an alternative simple defect, the singular fiber which is a sphere with one transversal
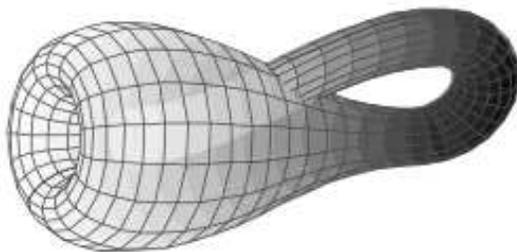
Figure 10: Representation of a Klein bottle in a 3D-space as a surface with self-intersection.

negative self-intersection point [39]. The objection to appearance in Hamiltonian systems of such a topologically allowed defect comes from the deformation arguments. Supposing that the "negative" defect is an elementary one, the small deformation of this singular fiber removing the singular point should lead to a fiber from the neighborhood filled by regular fibers. At the same time, a small deformation of a singular fiber which is a sphere with one transversal negative self-intersection point should lead to a regular fiber which is a non-orientable surface (Klein bottle) instead of a regular torus.

In order to see why this non-orientable surface cannot be another non-orientable surface, for example real projective plane RP_2, rather than the Klein bottle we can study what happens with a pinched RP_2 if we cut it through the pinch point. One can verify that a singly pinched RP_2, after cutting through the pinch point, becomes again RP^2 rather then sphere.

In contrast, if we prepare a singly pinched Klein bottle and cut it through the pinch point, the result may be different. It is important to note that the Klein bottle can be pinched in two non-equivalent ways: either by shrinking to zero the cycle which is a generator of the infinite group, or by shrinking to zero a generator of a Z_2 group. Cutting a pinched Klein bottle through the pinch point which corresponds to a vanishing cycle being a Z_2 generator leads to a 2D-sphere, S_2. At the same time cutting pinched Klein bottle through the pinch point which corresponds to a vanishing cycle being a Z generator of a group of integers results in an RP_2 surface. Thus appearance of a sphere with one negative self-intersection point as a generic singular fiber assumes that regular fibers should be Klein bottles rather than regular tori and the singularity should be associated with the vanishing of the Z_2 generator. It is probably useful to note here that the Klein bottle itself can be considered as a critical fiber of toric fibration associated with the double covering of a Klein bottle by a torus. Although a Klein bottle is known to be a generic critical fiber for toric foliations this critical fiber rarely appears in applications [5]. What kind of integrable systems can lead to generic fibers of Klein bottle type remains an open problem.
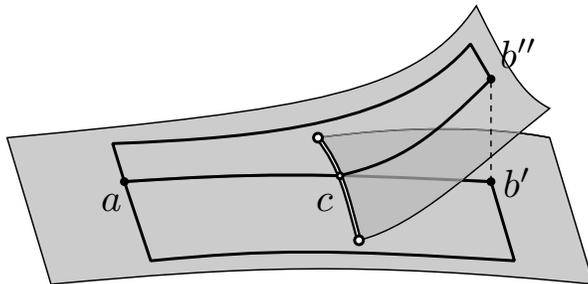
15

Figure 11: Self overlapping of a lower cell of the unfolding image of the energy-momentum map.

5 Multi-component energy momentum map, bidromy and so on

Another generalization of monodromy is based on the analysis of images of energy-momentum maps with several components. A rather complicated example of multicomponent maps arises even for such a naturally simple integrable model as a Manakov top [57]. A much more simpler example of the appearance of a second component was shown in Figure 1 c, where the second component appears as a result of a Hamiltonian Hopf bifurcation [19, 20] leading to transformation of an isolated focus-focus singular point into a second leaf attached to the main leaf through a family of singular fibers, named bitori (see Figure 2 c). Such creation of the second leaf evidently cannot modify the nontrivial monodromy associated with an initial isolated singular point and consequently the nonlocal monodromy is associated with the closed path surrounding the whole second leaf on the image of the energy-momentum map. More precisely, we should say that the closed path goes around bitorus stratum responsible for joining two components into one. Examples of such a transformation are well known in different molecular examples like a hydrogen atom in fields [20], or an LiCN [36, 22] molecule. It should be also noted that the appearance of the second leaf on the image of the energy-momentum map could also result from the fold type catastrophe [15]. In such a case the second component typically appears within the regular region of the image of the energy-momentum map and consequently the monodromy transformation associated with the closed path surrounding the second leaf should remain trivial in this case.

Another possibility of getting the second leaf appears in a rather different situation associated in fact with the self-overlapping of the same leaf (see Figure 1 d). The organization of the image of the energy momentum map can be explained by using a schematic "unfolding" procedure represented in Figure 11, which can be more accurately explained by introducing notions of upper and lower cells for singular toric fibrations as it is done in detail in [46].

The specificity of the situation is due to the fact that a certain region of

the image of the energy-momentum map has two regular tori as an inverse image (points b' and b'' in Figure 11). At the same time we can choose a continuous family of *regular* tori allowing deformation of one torus in this region into another. The possibility of such a deformation leads to an unambiguous definition of three tori lying close to a bitorus fiber (point c in Figure 11) on three locally different leaves of the image of the energy-momentum map. This construction gives the possibility to define the crossing of a bitorus line in such case. The tentative definition of corresponding splitting of cell and path when crossing the bitorus line and fusion of bipath into one path along with two cell fusion was named *bidromy* [54]. The bidromy phenomenon was initially illustrated on a rather complicated example of a three degree of freedom system with special resonance $1 : 1 : 2$ [30, 54] but essentially the same qualitative behavior can be observed for a 2-degree of freedom dynamical system. An example of such behavior was even found in such simple quantum systems as a hydrogen atom in the presence of electric and magnetic external fields [25, 24].

6   Time evolution and monodromy

Up until now I have discussed only a "static", in some sense, manifestation of Hamiltonian monodromy, namely special arrangements of joint spectrum of mutually commuting operators. A quite interesting direction of further investigation is related with the analysis of monodromy manifestations during time dependent processes. The general unprecise idea is to realize the time-dependent evolution of the dynamical system corresponding in some sense to going along a closed path surrounding a singularity in the energy-momentum space. The initial question in such a construction is about what should be observed and what fingerprints of monodromy could be found. The first step in this direction was made by Delos et al [13, 14] using very simple toy problem, namely the motion of a single particle in an axially symmetric billiard with a parabolic barrier potential. This problem possesses monodromy in its stationary Hamiltonian formulation. In order to see the nontrivial dynamic effect of monodromy, one needs to follow the evolution of a family of particles and to choose a special time-dependent perturbation which allows to change values of the integral of motion in, say, an adiabatic way. The dynamic manifestation of monodromy consists in the nontrivial topological modification of an initial spatial distribution of particles after following a closed path in the energy-momentum space and returning to the initial values of integrals of motion. The most difficult step in the realization of such a time-dependent processes is to find the precise form of the time dependent perturbation which satisfies all theoretically imposed assumptions on the form of perturbation and to realize it practically.

7   Perspectives

The notion of monodromy can be related to problems which are quite far from the classical Hamiltonian integrable systems, or model quantum molecules. The idea of such a generalization is based on the relation between defects of regular patterns and the monodromy. Namely, many defects of regular lattices which appear in solid state physics can be considered as a cumulative result of a number of elementary monodromy defects and can be treated as some complicated non-elementary defects from the point of view of monodromy defects. It is clear that the choice of "elementary bricks" is not unique. Even though the mathematical description of defects in solid state physics and in toric fibrations related to dynamical Hamiltonian systems, or in other models turns out to be similar, the relevance of these mathematical constructions should be confronted to physical reality.

From the physical point of view the origin of defects in solids is due to imperfection of the crystal growth. Some of 2D-point defects like disclinations has a natural description in terms of several elementary monodromy defects. Some others, like vacations, have nothing to do with monodromy, because they are not related to the topology of the lattice. We can try to generalize the mathematical description of defects in terms of elementary monodromy defects and to look from this point of view for typical singularities (defects) of almost regular patterns appearing in different domains. The general idea behind this is to find interpretation of defects in terms of natural "elementary ones" and to specify the generic most frequently appearing defects and to find a possible explanation of their appearance.

Regular patterns with defects can appear not only in solid state, with each point being associated with an atom or molecule, but in more complex systems like plants with regular patterns being associated with leaves or seeds, reflecting the morphogenesis or the plant development. The most striking example of such a regular pattern formation is the phyllotaxis, intriguing scientists working in different fields even quite far from biology. Let us just cite Leonardo da Vinci, Kepler, Bravais, Turing, Coxeter, .. [1].

The phenomenon of phyllotaxis describes the morphology of many botanical objects. It exists in the arrangement of repeated units such as leaves around a stem in various plants, seeds of a pine-cones or of a sunflower, scales of a pineapple, etc. The most widely known is the spiral phyllotaxis associated in a major part of cases with lattices formed by left hand and right hand spirals whose number are found to be consecutive numbers in the Fibonacci sequence $1, 1, 2, 3, 5, 8, ..., a_k, a_{k+1}, a_{k+2} = a_k + a_{k+1}, ...$ The interdisciplinary character of the phyllotaxis phenomenon is clearly seen on the example of pattern formation by drops of ferro-fluids in a magnetic field [16] or by flux lattices in superconductors [38].

The enormous literature devoted to the study and to the explanation of the universal behavior of botanical patterns mainly deals with a peculiar presence of Fibonacci numbers and chemical regulation of their presence (see [34, 64, 1, 50, 35, 58] and references therein). Characterization of the resulting pattern from

Figure 12: Sunflower with 55 right spirals (parastichies) indicated by additional lines to guide the eyes.

Figure 13: The same sunflower as in Figure 12, but now 34 left spirals are indicated.
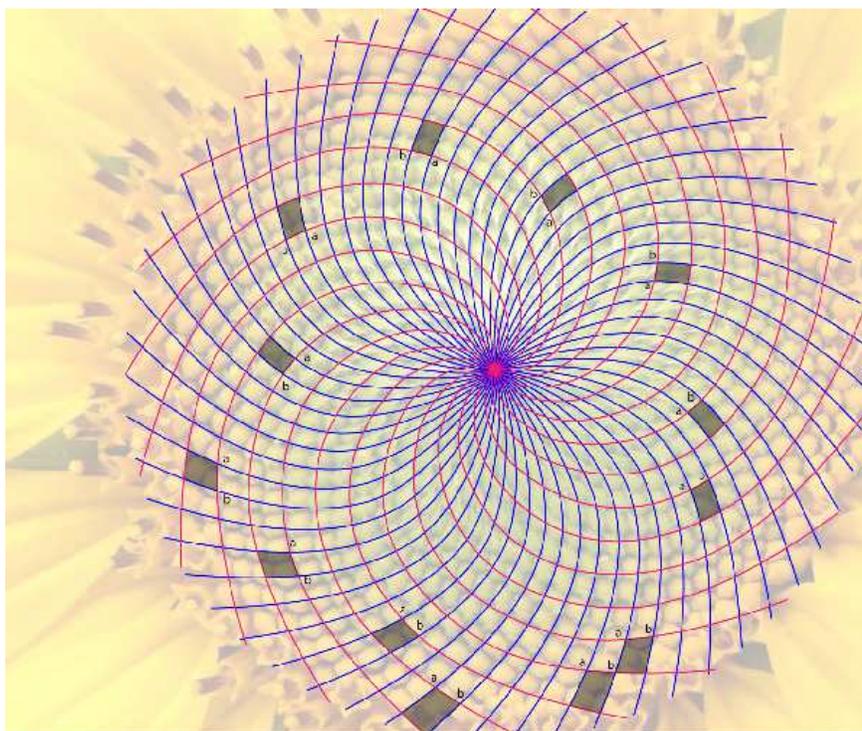
Figure 14: "Sunflower lattice" formed by left and right spirals shown in previous figures 12, 13. The transformation of an elementary cell of this lattice along the close path surrounding the central singularity shows the presence of monodromy, related to a 2π self-rotation of the elementary cell (see the text).

the point of view of the singularity responsible for the pattern formation and its monodromy has not been described earlier in the literature up to our knowledge. The appearance of defects within the spiral pattern and modifications of (n_1, n_2) into (n'_1, n'_2) patterns has been discussed on several occasions [56], but the most essential persistent feature of global organization of spirals due to the characteristic relation between a locally regular lattice and its nontrivial behavior around the growing center has not been related to a monodromy like notion.

I would like here to demonstrate the manifestation of a phyllotaxis monodromy on the sunflower example and to formulate several questions about universality of the patterns, associated defects, and relevance of such defects to evolution processes not only in botanic, but in other fields of science.

First of all, taking an example of sunflower with 34 left and 55 right eye-guided spirals - parastichies in biological language - (see Figure 12) we note that locally regular lattice can be constructed by explicitly plotting spirals (see Figure 13). In any local simply connected region this lattice can be easily transformed into a regular lattice but globally it has an easily seen defect - the apex. In order to see the nontrivial effect of the apex region on the lattice we take the elementary cell in any regular part of the lattice and move it step by step along a closed path surrounding the apex region (see Figure 14). At each step the local regular structure allows us to move the elementary cell unambiguously, even though the lattice itself and the corresponding elementary cell could be chosen in a different way because the basis of the lattice is defined as usual up to an $SL(2, Z)$ similarity transformation[1]. It is important to keep the vertices of the cell labeled at each step of the lattice displacement. We use letters a and b in Figure 14.

It is easy to verify that after following a closed path which does not go around the apex the elementary cell returns to its initial position. Apparently the same situation occurs after going along a closed path surrounding once the apex. But the principal difference is that in this case (shown in Figure 14) the cell returns to its original position after making a 2π self-rotation around axis passing through the center of the cell in an orthogonal to the cell-plane direction. From the point of view of Hamiltonian monodromy the comparison of the initial cell and the final cell can be expressed by an identity matrix which is associated with a trivial monodromy. At the same time the closed path is evidently non-contractible and in order to characterize the singularity (or the defect) responsible for this non-triviality we need to add another topological invariant associated with the closed path, namely the self-rotation number which can be positive, negative or zero. According to the earlier formulated correspondence between a cell transportation around a defect and the "monodromy defect" construction (by removing or inserting a wedge) the positive numbers of self rotation correspond to defects with removing wedge. These defects correspond to typical singularities (focus-focus) for Hamiltonian systems. The direction of

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[1]The corresponding ambiguity in the choice of parastichies was equally mentioned in literature discussing the geometrical aspects of phyllotaxis [7]

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self-rotation of the cell is well defined for an arbitrary number of focus-focus singularities in Hamiltonian systems [11, 12, 66]. This direction is related to the fact that the corresponding defect is produced by removing a certain number of wedges from the lattice. It is important to note that the direction of "positive" self-rotation should be defined only after specifying the direction of the closed path going around the singularity. The rotation can be defined positive if the cell turns from the $\mathbf{v}_1$ to $\mathbf{v}_2$ in the shortest way, where $\mathbf{v}_1$ is a vector defining the direction of the path surrounding the singularity, and $\mathbf{v}_2$ is a vector joining the cell and the singularity. More formally, the triple product $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{R}) > 0$ should be positive, where $\mathbf{R}$ is the axial vector giving the self-rotation of the cell.

In a more intuitive way we can say that the positive choice of the cell's self-rotation coincides with the sense of rotation of a wheel (representing the cell) turning around another wheel (representing the singularity), if the "cell" rolls around the "singularity" without slipping. Unfortunately, in order to get in such a representation the value of the self-rotation angle to be equal to 2π, the radius of a wheel representing the singularity should be chosen to be equal to 0.

The choice of the sign of the monodromy defect observed for a wide range of botanic patterns seems to be rather fundamental property similar in the spirit to the left-right asymmetry and time irreversibility. A number of interesting questions naturally arise provoked by this supposition. Can the evolution of the plants be modeled by a dynamical system with the source being associated with a generic singularity characterized by an identity monodromy and positive 2π self-rotation? Does the sign of self-rotation reflect specific properties of the system? We end with an even more general tentative speculation: Is the generic singularity associated with irreversible time evolution (growing process) always characterized by a trivial (identity) monodromy with positive 2π self-rotation? Is it possible to apply this conjecture to the evolution of our Universe?

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