Chern number modification in crossing the boundary between different band structures. Three-band models with cubic symmetry

T. Iwai[a] and B. Zhilinskii[b]

[a] Kyoto University, 606-8501 Kyoto, Japan,
iwai@amp.i.kyoto-u.ac.jp
[b] Université du Littoral Côte d'Opale, 59140 Dunkerque, France,
boris@purple.univ-littoral.fr

Abstract

A family of $n \times n$ Hermitian matrix Hamiltonians defined on the sphere S^2 and depending on extra control parameters in the presence of a finite subgroup of $SO(3)$ as a symmetry group are studied with eigen-line bundles which are constructed by piecing together locally-defined eigenvectors. The condition for degeneracy in eigenvalues splits in general the space of control parameters into distinct iso-Chern domains on each of which the Chern numbers of the associated eigen-line bundles are constant. A Chern number modification or a delta-Chern occurs when crossing the boundary from one iso-Chern domain to another. The present article provides a formula for the delta-Chern on the model of two-parameter family of 3×3 Hermitian matrix Hamiltonians with cubic symmetry together with the whole sets of Chern numbers on respective iso-Chern domains.

Keywords: Energy bands, Chern numbers, delta-Chern or Chern number modification, cubic symmetry.

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1 Introduction

Topology in condensed matter physics, especially in topological insulators has drawn great interest in recent years [1, 2]. In contrast to this, topology in molecular physics [3] has gained less attention. From the viewpoint of mathematical physics, both theories have share the same conceptual grounds, one of which is energy band rearrangement. The present article with its root in the molecular physics studies intensively Chern number modification, which is closely associated with energy band rearrangement, and shares topological techniques with the topological insulator theory. This article is directly a successor to authors' papers [4, 5, 6]. As physical ideas for Chern number modification are described in detail in [6], this paper concentrates on a mathematical analysis of Chern number modification.

From a historical viewpoint, the study of a parameter family of Hamiltonians started with [7] for the redistribution of energy levels against a change in the parameter. In the same line of study, a relation between monodromy and band rearrangement was discussed in [8] and its quantum manifestation was treated in [9]. Analysis of physical systems in terms of vector bundles and associated Chern numbers, as it is done in the present article, has been widely used in other branches of physics. For example, in the study of the quantum Hall effect [10], a complex line bundle over the two-torus as a Brillouin zone is naturally introduced together with a Berry connection in terms of Bloch wave functions, and the technique for evaluating the Chern number of the line bundle is similar to that to be used in the present article. Interesting topological phenomena in electronic energy bands are the quantum spin Hall effect and topological insulators with time-reversal symmetry (see [11, 12, 13] for example, a few of a huge number of references). Though the interest of this article originates from rearrangements of molecular energy bands, the bundle picture used in this article shares the same mathematical grounds as these studies. The results obtained in this article allows natural physical interpretation as well. For example, the fact that Chern numbers are constant on an iso-Chern domain means that the topological property of the physical system in question is robust against a small change in control parameters, and the fact that a Chern number modification or a delta-Chern occurring only when crossing a degeneracy curve in the control parameter space is interpreted as a drastic change in physical property accompanying an energy gap closing. The delta-Chern formula to be studied in this article describes this qualitative change, and would be called a kind of "wall-crossing" formula [14] in the sense of basic idea. The mathematical foundations of studies concerning a family of Hermitian matrices are described in [15]. In addition, though the delta-Chern is authors' nomenclature, a similar idea was given in [16] and in [17], in the latter of which the curvature is treated in the sense of distribution when degeneracy in eigenvalues takes place.

In this article, eigen-lines bundles are dealt with for a family of $n \times n$ Hermitian matrix Hamiltonians defined on the sphere S^2 as the space of classical dynamical variables and depending on extra control parameters in the presence of a finite subgroup of $SO(3)$ as a symmetry group, where each of the eigen-line bundles is formed by piecing together locally-defined eigenvectors associated with each of the eigenvalues of the matrix Hamiltonian, if eigenvalues are not degenerate. Let $\mathbb{R}^m$ denote the space of control parameters and G a symmetry group. The product space $\mathbb{R}^m \times S^2$ is considered as the total parameter space for the family of the Hamiltonians with symmetry. The distinction between dynamical variables and control parameters is made by the action of the symmetry group; the symmetry group G acts on S^2 but not on $\mathbb{R}^m$. The condition for degeneracy in eigenvalues splits in general the space of control parameters $\mathbb{R}^m$ into disjoint "iso-Chern" domains. This is because the codimension of degeneracy in eigenvalues of Hermitian matrices is three [18] whereas $\dim S^2 = 2$ and $\dim \mathbb{R}^m = m$. In correspondence to each point of respective boundaries (or degeneracy surfaces) of the iso-Chern domains in $\mathbb{R}^m$, there appear on S^2 isolated degeneracy points at which eigenvalues are degenerate. At each point of a domain bounded by degeneracy surfaces, there are assigned eigen-line bundles associated with respective non-degenerate eigenvalues. Since the Chern numbers of respective eigen-line bundles are constant on each domain in $\mathbb{R}^m$, the domains are called iso-Chern domains. For each iso-Chern domain, the trivial bundle $\mathbb{C}^n \times S^2$ over S^2 is decomposed into the direct sum of eigen-line bundles, each of which is characterized by a constant Chern number.

The interest of the present article centers on a Chern number modification or a delta-Chern which accompanies the crossing of the boundary from one iso-Chern domain to another along a path transverse to the boundary. The delta-Chern can be understood as describing a rearrangement of band structures. Though the eigen-line bundles cannot be defined for degeneracy points, a close look at the neighborhood of a degeneracy point together with eigenvectors can reveal how transition goes on in eigen-line bundle structures against a change in control parameters along a path crossing the boundary in the control parameter space. It is to be noted here that the degeneracy point is referred to in two ways; one is a degeneracy point in the control parameter space and the other in S^2. However, in order to observe the transition in question, the degeneracy point is to be considered as a pair of the degeneracy points mentioned above or as a point of $\mathbb{R}^m \times S^2$. Then, the full Hamiltonian should be projected, after being linearized at the degeneracy point, onto a matrix acting on the eigenspace associated with the doubly degenerate eigenvalues in general. If linearization fails, the first non-vanishing higher-order terms of the local expansion of the full Hamiltonian should be adopted. The resultant Hamiltonian is called a local Hamiltonian, which is 2×2 Hermitian matrix with entries being linear in suitably-defined local coordinates or with entries consisting of first non-vanishing higher-order terms. Since the full Hamiltonian has the G symmetry, the local Hamiltonian has a local symmetry as well, for which the symmetry group, called a local symmetry group, is the isotropy subgroup G_0 of G at the degeneracy point in S^2. The isotropy subgroup G_0 is represented in the two-dimensional eigenspace associated with the doubly degenerate eigenvalues. Since G_0 is a subgroup of $SO(2)$, it is an Abelian group, so that it is reducible. Then, there exists a basis in the eigenspace in question such that the G_0 is represented in the diagonal form. The local Hamiltonian expressed with respect to such a basis is called being in normal form. From the eigenvectors of the local Hamiltonian in normal form, one can extract a necessary quantity for the delta-Chern. To be precise, such a quantity is a contribution from a single degeneracy point for the delta-Chern, which is referred to as a local delta-Chern. The local delta-Chern is exact in spite of the fact that only first non-vanishing terms of the local expansion of the full Hamiltonian are used. This is because the local delta-Chern can be put in the form of integer-valued contour integral and because the integrand and the contour can be deformed without changing the resultant integer value. Because of the full symmetry group G, the local Hamiltonian on the neighborhood of a degeneracy point is equivalent to those at the other degeneracy points if the degeneracy point in S^2 is in the same orbit of G. This means that the local delta-Chern is the same throughout the orbit, so that the global delta-Chern is the product of the local delta-Chern by the order of the orbit in question. The local delta-Chern depends on the direction in which the boundary is crossed, and hence it should be calculated with special carefulness in its sign.

The strategy for finding a local delta-Chern has several steps. The first step is to find both the degeneracy surfaces (or the boundaries of iso-Chern domains) in the space of control parameters and the degeneracy points in S^2. According to the G action, the sphere S^2 is stratified into strata, where at every point of each stratum, the isotropy subgroup is the same up to conjugation. Since the degeneracy points in S^2 form an orbit of the symmetry group G, one can start in principle by picking up a degeneracy point from different 0-dimensional strata, then proceed to a degeneracy point from 1-dimensional stratum, and finally from 2-dimensional (generic) stratum. In a case, one is allowed to show the absence of degeneracy points at strata other than 0-dimensional one. Evaluating the full Hamiltonian at respective degeneracy points in S^2, and calculating its eigenvalues or using the relevant discriminant, one can determine degeneracy curves in the control parameter space. If G is the octahedral group, the stratification of the sphere S^2 is well known, and this step is rather easy to perform.

The next step is to form a local Hamiltonian in normal form in the neighborhood of a degeneracy point. Tasks to do are (i) to draw a short curve in the control parameter space, which crosses transversely a degeneracy surface at a regular point of a degeneracy surface, *i.e.*, at a point other than intersections of two or more degeneracy surfaces, (ii) to take a tangent line to the curve in the control parameter space, (iii) to take a frame on the tangent plane to S^2 at the degeneracy point concerned, with respect to which a local coordinates are defined, (iv) to choose a basis of the eigenspace associated with the doubly degenerate eigenvalues, with respect to which the isotropy subgroup at the degeneracy point is represented in the diagonal form, (iv) to find a local Hamiltonian by linearization or by taking the first non-vanishing higher-order terms, which is defined on the product of the tangent line and the tangent plane in question and described in terms of local coordinates defined there. The resultant local Hamiltonian has local symmetry and is in normal form.

The final step is to determine a local delta-Chern by using the local Hamiltonian in normal form, a formula for which will be shown in the subsequent sections. The local delta-Chern is shown to be independent of the choice of bases and the choice of frames. The total (or global) delta-Chern is now easy to obtain, which is $\#G/\#G_0$ times the local delta-Chern, where $\#G$ and $\#G_0$ denote the numbers of elements of G and of G_0, respectively.

To perform the above-mentioned procedure, model Hamiltonians are adopted from [19, 20], which are 2×2 and 3×3 matrix Hamiltonians, respectively, admitting symmetry by the octahedral group. The Hamiltonians adopted are not so simple but can be manipulated in search of delta-Cherns. In [19, 20], energy band rearrangements were qualitatively studied on the level of classical limit for slow subsystem, but topological invariants associated with the semi-quantum description of coupled "slow" and "fast" subsystems were not introduced at that time. In [4], Chern number modification is treated for the 2×2 Hermitian matrix Hamiltonians with $SO(2)$ and with D_3 symmetries. In the study of these Hamiltonians, delta-Chern was not used, since it was possible to obtain Chern numbers on respective iso-Chern domains in a straightforward manner. It is to be noted in addition that as far as rotation-vibration states near an equilibrium are concerned, the entries of a model Hamiltonian may be restricted to polynomials on $\mathbb{R}^3$. Further, if the Hamiltonian is required to be invariant under a discrete or a continuous subgroup, polynomials are chosen according to some of representations of the subgroup.

The organization of this article is as follows: Section 2 is a brief review of symmetry of the Hamiltonian with interest in the degeneracy in eigenvalues. Symmetry of the linearized Hamiltonian is also treated. In Sec. 3, the octahedral group is briefly reviewed in relation to the symmetry of two- and three-level model Hamiltonians (*i.e.*, $n = 2, 3$). More details will be given in Appendix. Section 4 is concerned with eigen-line bundles associated with eigenvalues of a generic 2×2 matrix Hamiltonian. Chern numbers are not touched, which will be treated in the succeeding sections with concrete examples. In Sec. 5, a family of two-level model Hamiltonian with O symmetry is studied by calculating the Chern numbers assigned to respective iso-Chern domains, during which different approaches are explicitly used for calculating the Chern numbers. In Sec. 6, the linearization of the full Hamiltonian and its retraction to a two-level model Hamiltonian are discussed explicitly in the case of the three-level models. The local Hamiltonian in normal form is discussed as well. Section 7 is concerned with 2×2 local Hamiltonians in normal form with interest in relevant invariants. In Sec. 8, after studying eigen-line bundles for the 2×2 local Hamiltonians in normal form, a formula for the local delta-Chern is proved. In Sec. 9, a formula for the global delta-Chern is obtained by means of homotopic deformation of integer-valued contour integrals. In Sec. 10, a family of three-level model with O symmetry is worked out by applying the global delta-Chern formula to obtain an iso-Chern diagram, in which a column of Chern numbers are assigned to each iso-Chern domain. The results obtained after an intricate and long calculation are summarized in Figs. 6 and 7, which have been announced in [6]. In Sec. 11, the two-level model treated in Sec. 5 is revisited from the viewpoint of the delta-Chern shown in Sec. 8. Section 12 contains a case study to observe a change in Chern numbers in the case of triple degenerate eigenvalues on an explicit model Hamiltonian. In Section 13, linear approximation method is discussed in a generic situation together with a relation of the local-Chern analysis to the Berry phase. Section 14 offers remarks on possible extensions of the delta-Chern analysis to the cases where the linear approximation fails but quadratic approximation works and where a triple degeneracy occurs in eigenvalues. Section 15 contains concluding remarks on interesting relations between possible values of Chern numbers and the decomposition rules for tensor products of group representations. In addition, a progress achieved after the present work is touched upon. Appendix contains a review of the octahedral group together with the symmetry conditions for the Hamiltonians.

2 Symmetry of the Hamiltonian

As is anticipated in Introduction, we have to discuss the symmetry of the Hamiltonian from two points of view, local and global. For the sake of contrast, we refer to the symmetry by the full group as the global symmetry and that by the isotropy subgroup at a point of S^2 as the local symmetry.

Let G be an discrete subgroup of $SO(3)$ acting on $S^2 \subset \mathbb{R}^3$. A Hamiltonian $H(\boldsymbol{x}), \boldsymbol{x} \in S^2$, an $n \times n$ Hermitian matrix defined on S^2, admits the symmetry group G if and only if

$$H(g\boldsymbol{x}) = D(g)H(\boldsymbol{x})D(g)^{-1}, \quad g \in G, \tag{1}$$

where $D(g)$ is a representation matrix acting on $\mathbb{C}^n$, and where $H(\boldsymbol{x})$ may depend on extra control parameters.

Since the codimension for degeneracy in eigenvalues of Hermitian matrices is three and since the dimension of S^2 is two, degeneracy points may appear on a zero-dimensional subset of S^2, in general. If λ_0 is an eigenvalue of $H(\boldsymbol{x})$ doubly degenerate at $\boldsymbol{x}_0$, one has

$$\det(H(\boldsymbol{x}_0) - \lambda I)|_{\lambda=\lambda_0} = \frac{d}{d\lambda} \det(H(\boldsymbol{x}_0) - \lambda I)\bigg|_{\lambda=\lambda_0} = 0. \tag{2}$$

Equations (1) and (2) are put together to give

$$\det(H(g\boldsymbol{x}_0) - \lambda I)|_{\lambda=\lambda_0} = \frac{d}{d\lambda}\det(H(g\boldsymbol{x}_0) - \lambda I)\bigg|_{\lambda=\lambda_0} = 0. \tag{3}$$

This implies that if $\boldsymbol{x}_0 \in S^2$ is a degeneracy point, the orbit of $\boldsymbol{x}_0$ by G is a set of degeneracy points.

The symmetry for H can be extended to that for the derivative of H. For the time being, we consider $H(\boldsymbol{x})$ as defined on $\mathbb{R}^3$. Let us introduce the variables $\boldsymbol{x}'$ by $\boldsymbol{x}' = g\boldsymbol{x}$. In components, we put it in the form $x'_j = \sum g_{jk}x_k$. We now compute the gradient, $\nabla H(\boldsymbol{x}') = \left(\frac{\partial h_{ij}(\boldsymbol{x}')}{\partial x'_k}\right)$, of H, where the symbol ∇H takes values in the tensor product space $\mathbb{R}^3 \otimes \mathbb{C}^{n\times n}$. Then, the symmetry condition $H(g\boldsymbol{x}) = \mathrm{Ad}_{D(g)}H(\boldsymbol{x})$ is differentiated to give

$$\frac{\partial h_{ij}(\boldsymbol{x}')}{\partial x'_k} = \sum_\ell g_{k\ell}\sum_{p,q} D(g)_{ip}\frac{\partial h_{pq}(\boldsymbol{x})}{\partial x_\ell}\overline{D(g)}_{jq}, \tag{4}$$

which is compactly written as

$$\nabla H(g\boldsymbol{x}) = (g \otimes \mathrm{Ad}_{D(g)})\nabla H(\boldsymbol{x}), \tag{5}$$

where $\nabla H(\boldsymbol{x}) = (\frac{\partial h_{ij}}{\partial x_k})$ takes values in $\mathbb{R}^3 \otimes \mathbb{C}^{n\times n}$ and where the action of $g \otimes \mathrm{Ad}_{D(g)}$ are defined on $\mathbb{R}^3 \otimes \mathbb{C}^{n\times n}$ by

$$\boldsymbol{a} \otimes A \mapsto (g \otimes \mathrm{Ad}_{D(g)})(\boldsymbol{a} \otimes A) = g\boldsymbol{a} \otimes \mathrm{Ad}_{D(g)}A, \quad \boldsymbol{a} \in \mathbb{R}^3,\ A \in \mathbb{C}^{n\times n}. \tag{6}$$

We return to the initial Hamiltonian defined on S^2. We are to show that the invariance condition (1) is naturally extended to that for the derivative of the Hamiltonian. Let $\Pi_{\boldsymbol{x}}$ denote the tangent plane to S^2 at a point $\boldsymbol{x} \in S^2$, and $\boldsymbol{\xi}_k, k = 1, 2$, be an orthonormal system of tangent vectors with positive orientation on $\Pi_{\boldsymbol{x}}$. The Cartesian coordinates q_k are defined on $\Pi_{\boldsymbol{x}}$ through $\sum q_k\boldsymbol{\xi}_k$. The homogeneous linear Hamiltonian is defined to be

$$H_1(q;\boldsymbol{x}) = \sum q_k\boldsymbol{\xi}_k \cdot \nabla H(\boldsymbol{x}), \tag{7}$$

where $\boldsymbol{\xi}_k \cdot \nabla H(\boldsymbol{x})$ denotes the inner product $\mathbb{R}^3 \times (\mathbb{R}^3 \otimes \mathbb{C}^{n\times n}) \to \mathbb{C}^{n\times n}$, or the $n \times n$ complex matrix with (ℓ, m) entries $\boldsymbol{\xi}_k \cdot \nabla h_{\ell m}$.

From (5) and (7), it follows that if the frame $g\boldsymbol{\xi}_k$ is chosen at $g\boldsymbol{x}$, the $H_1(q;\boldsymbol{x})$ is subject to the transformation

$$H_1(q; g\boldsymbol{x}) = \mathrm{Ad}_{D(g)}H_1(q;\boldsymbol{x}), \tag{8}$$

where use has been made of the fact that g is an orthogonal transformation.

If $g_1\boldsymbol{x} = g\boldsymbol{x}$, we set $h = g_1g^{-1}$, which is an element of the isotropy subgroup at $g\boldsymbol{x}$. Two frames $g\boldsymbol{\xi}_k$ and $g_1\boldsymbol{\xi}_k$ on the tangent plane at $g_1\boldsymbol{x} = g\boldsymbol{x}$ are related by

$$g_1\boldsymbol{\xi}_k = hg\boldsymbol{\xi}_k = \sum h^{(2)}_{jk}g\boldsymbol{\xi}_j, \tag{9}$$

where $h^{(2)}$ denotes the representation matrix of h with respect to the frame $g\boldsymbol{\xi}_k$. On account of this, the $H_1(q;\boldsymbol{x})$ transforms according to

$$\mathrm{Ad}_{D(g_1)} H_1(q;\boldsymbol{x}) = H_1(h^{(2)}q; g_1\boldsymbol{x}). \tag{10}$$

In particular, if $g_1\boldsymbol{x} = g\boldsymbol{x} = \boldsymbol{x}$, this equation becomes

$$\mathrm{Ad}_{D(h)} H_1(q;\boldsymbol{x}) = H_1(h^{(2)}q; \boldsymbol{x}), \quad h \in G_{\boldsymbol{x}}, \tag{11}$$

which describe the local symmetry of the linearized Hamiltonian $H_1(q;\boldsymbol{x})$ with respect to the isotropy subgroup $G_{\boldsymbol{x}}$ at $\boldsymbol{x}$.

3 Symmetry by the octahedral group O

The octahedral group O is the orientation-preserving symmetry group for the regular octahedron, which is known to be isomorphic to the symmetric group S_4 and further to be generated by

$$C_4^Z \mapsto \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \quad C_3^{[-1-1-1]} \mapsto \begin{pmatrix} & & 1 \\ & & & \\ 1 & & \end{pmatrix} \tag{12}$$

to form a discrete subgroup of $SO(3)$. See Appendix for symbols C_4^Z and $C_3^{[-1-1-1]}$ and for further details. The two-dimensional representation E of the O group is known to be generated by

$$C_4^Z \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad C_3^{[-1-1-1]} \mapsto \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}. \tag{13}$$

Let $\mathcal{H}_0(2)$ and $\mathcal{H}_0(3)$ denote the sets of traceless 2×2 and 3×3 Hermitian matrices, respectively. The model Hamiltonian we treat in this article takes values in $\mathcal{H}_0(2)$ or $\mathcal{H}_0(3)$, according to whether the semi-quantum system in question is associated with two- or three-level model. From the view point of symmetry we are interested in at present, $\mathcal{H}_0(2)$ and $\mathcal{H}_0(3)$ are decomposed into

$$\mathcal{H}_0(2) = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & ic \\ -ic & 0 \end{pmatrix} \right\} \tag{14}$$

and

$$\mathcal{H}_0(3) = \left\{ \begin{pmatrix} 0 & c_1 & b_1 \\ c_1 & 0 & a_1 \\ b_1 & a_1 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & -ic_2 & ib_2 \\ ic_2 & 0 & -ia_2 \\ -ib_2 & ia_2 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \right\}, \tag{15}$$

respectively, where a, b, c, a_k, b_k, c_k, and d_j are real parameters with $d_1+d_2+d_3 = 0$. Each subspace of $\mathcal{H}_0(2)$ and of $\mathcal{H}_0(3)$ carries an irreducible representation of the O group.

On the other hand, the sets of functions

$$\{2z^2 - x^2 - y^2,\ \sqrt{3}(x^2 - y^2)\} \quad \text{and} \quad xyz \tag{16}$$

are known as the basis of the E-representation and of the A_2-representation of the O group, respectively. Further, the sets of functions

$$\{x,\, y\; z\}, \quad \{yz,\, zx,\, xy\}, \quad \text{and} \quad \{x(x^2-\frac{3}{5}r^2), y(y^2-\frac{3}{5}r^2), z(z^2-\frac{3}{5}r^2)\} \tag{17}$$

are known to be bases of the T_1, T_2 and T_1 representations, respectively.

In view of these facts, we mainly treat the Hamiltonians

$$H(\boldsymbol{x}) = \begin{pmatrix} a(2z^2-x^2-y^2) & \sqrt{3}a(x^2-y^2)-ibxyz \\ \sqrt{3}a(x^2-y^2)+ibxyz & -a(2z^2-x^2-y^2) \end{pmatrix} \tag{18}$$

and

$$\begin{aligned} H(\boldsymbol{x}) \;&=\; \begin{pmatrix} 0 & iz & -iy \\ -iz & 0 & ix \\ iy & -ix & 0 \end{pmatrix} \\ &+a \begin{pmatrix} y^2+z^2-2x^2 & 0 & 0 \\ 0 & z^2+x^2-2y^2 & 0 \\ 0 & 0 & x^2+y^2-2z^2 \end{pmatrix} \\ &+b \begin{pmatrix} 0 & xy & zx \\ xy & 0 & yz \\ zx & yz & 0 \end{pmatrix}, \end{aligned} \tag{19}$$

which satisfy the invariance condition (1) with due representation matrix $D(g)$, where a, b are real parameters and where g acting on S^2 is viewed as subject to the T_1 representation. For (18), we pose the condition $(a, b) \neq (0, 0)$. Three-level model Hamiltonians depending on polynomials of degree three will be discussed on the occasion of necessity.

It is to be noted that the decompositions (14) and (15) are not the only possible decompositions but there are also other decompositions with respect to irreducible representations of the O group. For example, the first component subspace in the right-hand side of (14) is further decomposed into the direct sum of one-dimensional subspaces which carry A_1 or A_2 representation of the O group. Such cases have been already treated in [5].

4 Setting up eigen-line bundles

We consider the 2×2 Hermitian matrix

$$H^{(2)} = \begin{pmatrix} a_{11} & c_{12} \\ \overline{c}_{12} & a_{22} \end{pmatrix}, \tag{20}$$

where a_{11} and a_{22} are real-valued, and c_{12} is complex-valued. The domain on which these elements are defined is either the unit sphere S^2 or the tangent plane to the sphere at an assigned point of S^2. We will treat both cases in succeeding sections.

The eigenvalues of this matrix are given by

$$\lambda^{\pm} = \frac{1}{2}\Big(a_{11} + a_{22} \pm \sqrt{(a_{11}-a_{22})^2 + 4|c_{12}|^2}\Big). \tag{21}$$

For each of the distinct eigenvalues $\lambda^{\pm}$, we have two expressions of the normalized eigenvectors associated with $\lambda^{\pm}$;

$$|u^{\pm}_{\text{up}}\rangle = \frac{1}{\sqrt{(\lambda^{\pm} - a_{11})^2 + |c_{12}|^2}} \begin{pmatrix} c_{12} \\ \lambda^{\pm} - a_{11} \end{pmatrix}, \tag{22a}$$

$$|u^{\pm}_{\text{down}}\rangle = \frac{1}{\sqrt{(\lambda^{\pm} - a_{22})^2 + |c_{12}|^2}} \begin{pmatrix} \lambda^{\pm} - a_{22} \\ \bar{c}_{12} \end{pmatrix}, \tag{22b}$$

where the "up" and "down" eigenvectors are formed from the first and the last lines of the eigenvalue equation, respectively. Here, points at which the quantities inside the square root symbols vanish are called exceptional points and hence the domain $U^{\pm}_{\text{up/down}}$ of each eigenvector is the sphere with the assigned exceptional points removed. A straightforward calculation shows that the exceptional points for the "up" eigenvector associated with the larger eigenvalue λ^{+} are determined by

$$a_{11} - a_{22} > 0, \quad c_{12} = 0, \tag{23}$$

and those for the "down" eigenvector by

$$a_{11} - a_{22} < 0, \quad c_{12} = 0. \tag{24}$$

For the smaller eigenvalue λ^{-}, the exceptional points for the "up" and "down" eigenvectors are determined by (24) and by (23), respectively. Putting together these conditions, we have the table for the assignment of exceptional points,

	"up"	"down"
λ^{+}	$a_{11} - a_{22} > 0,\ c_{12} = 0$	$a_{11} - a_{22} < 0,\ c_{12} = 0$
λ^{-}	$a_{11} - a_{22} < 0,\ c_{12} = 0$	$a_{11} - a_{22} > 0,\ c_{12} = 0$

(25)

The "up" and "down" eigenvectors are related by

$$|u^{\pm}_{\text{up}}\rangle = \Phi^{\pm}|u^{\pm}_{\text{down}}\rangle \quad \text{on} \quad U^{\pm}_{\text{up}} \cap U^{\pm}_{\text{down}}, \tag{26}$$

where $\Phi^{\pm}$ are called transition functions, and are found to be expressed as

$$\Phi^{\pm} = \varepsilon^{\pm}\frac{c_{12}}{|c_{12}|}, \quad \varepsilon^{\pm} = \operatorname{sgn}(\lambda^{\pm} - a_{22}). \tag{27}$$

If we consider the matrix $H^{(2)}$ as defined on the unit sphere, Eq. (26) determines the complex line bundles associated with respective eigenvalues $\lambda^{\pm}$, which we call the eigen-line bundles. If the Hamiltonian $H^{(2)}$ is taken as defined on the tangent plane at a point $\boldsymbol{x} \in S^2$, we have eigen-line bundles over the tangent plane.

We don't give here the definition of the connection and the curvature for the eigen-line bundle, which will be given later in treating concrete models together with some techniques to evaluate the Chern number (see Sec. 5.1).

If we start with a 3×3 Hermitian matrix for a three-level model, we will have eigen-line bundles associated with each of non-degenerate eigenvalues in a similar manner. Here, the "up" eigenvector is formed by using the first and the second lines of the eigenvalue equation, and the "down" eigenvector by using the second and the last lines, for example. The transition function is defined in the same manner as in (26) and thereby an eigen-line bundle is determined. In Sec. 10.2, we will give a simple example for which the Chern number of the eigen-line bundles are easy to find. However, the eigenvalues are not easy to find in general, so that the domains $W_{\mathrm{up/down}}$ of "up" and "down" eigenvectors for each eigenvalue are not easy to identify either, and the explicit construction of the eigen-line bundle is difficult in practice. We then need to devise a method for resolving this difficulty, which will be shown in Secs. 6-9.

5 A two-level model with O symmetry

In this section, we treat the Hamiltonian (18) corresponding to the physical molecular model studied in [19] on the classical limit level. For notational simplicity, we express (18) as

$$H(\boldsymbol{x}) = \begin{pmatrix} a\phi_1 & a\phi_2 - ib\phi_3 \\ a\phi_2 + ib\phi_3 & -a\phi_1 \end{pmatrix}, \tag{28}$$

where

$$\phi_1 = 2z^2 - x^2 - y^2, \quad \phi_2 = \sqrt{3}(x^2 - y^2), \quad \phi_3 = xyz. \tag{29}$$

With this model, we show how to evaluate the Chern numbers of the eigen-line bundles.

5.1 Chern diagram

Proposition 5.1 The parameter space $\mathbb{R}^2 - \{0\}$ for the O-invariant Hamiltonian (28) retracts to a unit circle, and the degeneracy points on this circle are $(a, b) = (\pm1, 0), (0, \pm1)$. The Chern numbers c^+ (see (41)) of the eigen-line bundle associated with the positive eigenvalue are shown in Fig. 1, being assigned to respective iso-Chern domains (or arcs). The Chern number c^- of the eigen-line bundle associated with the negative eigenvalue is obtained by reversing the sign of the Chern number for the positive eigenvalue; $c^+ = -c^-$.

Proof. From (21) with $a_{22} = -a_{11}$, the condition of degeneracy in eigenvalues is described as

$$\det H(\boldsymbol{x}) = 0 \quad \Leftrightarrow \quad a^2(\phi_1^2 + \phi_2^2) = 0,\ b^2\phi_3^2 = 0. \tag{30}$$

Since the above condition is scale invariant, we may retract the control parameter space $\mathbb{R}^2 - \{0\}$ to the unit circle $a^2 + b^2 = 1$. There are four degeneracy points $(\pm1, 0)$, $(0, \pm1)$ on this circle, for which the eigenvalues of $H(\boldsymbol{x})$ are degenerate at certain points of S^2. For degeneracy points $(a, b) = (\pm1, 0)$ of the unit circle, the corresponding eight degeneracy points on the sphere S^2 are

$$\begin{pmatrix} \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \end{pmatrix}, \tag{31}$$

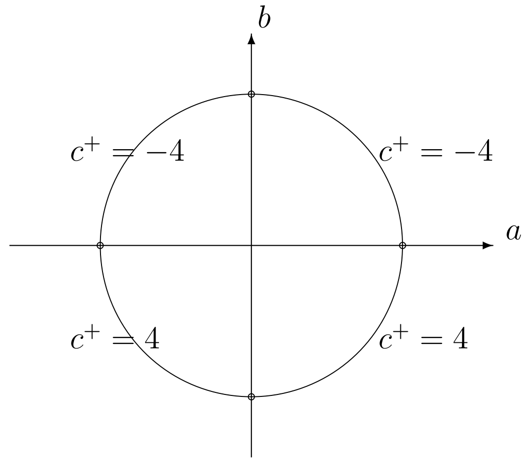

Figure 1: Chern numbers assigned to arcs of the unit circle

which form an orbit of the O group. The isotropy subgroup at each point of the orbit is isomorphic with C_3. For degeneracy points $(a, b) = (0, \pm 1)$, the set of corresponding degeneracy points on S^2 are given by

$$x^2 + y^2 = 1, \quad y^2 + z^2 = 1, \quad z^2 + x^2 = 1. \tag{32}$$

In these circles, there are six embedded degeneracy points

$$\begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}, \tag{33}$$

at which the isotropy subgroup is isomorphic to C_4, and further twelve embedded degeneracy points

$$\begin{pmatrix} \pm\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \pm\frac{1}{\sqrt{2}} \\ 0 \\ \pm\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \pm\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \end{pmatrix}, \tag{34}$$

at which the isotropy subgroup is isomorphic to C_2. The set of the above three circles without the orbits (33) and (34), which consists of 24 identical connected arcs, forms a set of degeneracy points at which the isotropy subgroup is isomorphic to C_1.

For regular values of the parameter, eigen-line bundles are associated with respective eigenvalues. From (23) and (24) with $a_{11} = a\phi_1 = -a_{22}$ and $c_{12} = a\phi_2 - ib\phi_3$, the exceptional points for the normalized eigenvector associated with the positive eigenvalue λ^+ are shown to be given by

$$\boldsymbol{n}_\pm = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}, \quad \boldsymbol{a}_\pm = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad \boldsymbol{b}_\pm = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}. \tag{35}$$

In the case of $a > 0$, the domains of normalized eigenvectors $|u^+_{\mathrm{up/down}}(\boldsymbol{x})\rangle$ associated with the positive eigenvalue λ^+ are

$$U^+_{\mathrm{up}} = S^2 - \{\boldsymbol{n}_\pm\}, \quad U^+_{\mathrm{down}} = S^2 - \{\boldsymbol{a}_\pm, \boldsymbol{b}_\pm\}, \tag{36}$$

respectively. The $|u^+_{\rm up/down}(\boldsymbol{x})\rangle$ are related by

$$|u^+_{\rm up}(\boldsymbol{x})\rangle = \Phi^+(\boldsymbol{x})|u^+_{\rm down}(\boldsymbol{x})\rangle, \quad \Phi^+(\boldsymbol{x}) = \frac{a\phi_2 - ib\phi_3}{\sqrt{a^2\phi^2 + b^2\phi_3^2}} \quad \text{on} \quad U^+_{\rm up} \cap U^+_{\rm down}. \tag{37}$$

The local connection forms are defined on respective domains to be

$$\omega^+_{\rm up} = \langle u^+_{\rm up}(\boldsymbol{x})|d|u^+_{\rm up}(\boldsymbol{x})\rangle, \quad \omega^+_{\rm down} = \langle u^+_{\rm down}(\boldsymbol{x})|d|u^+_{\rm down}(\boldsymbol{x})\rangle, \tag{38}$$

and related by

$$\omega^+_{\rm up} = (\Phi^+)^{-1}d\Phi^+ + \omega^+_{\rm down} \quad \text{on} \quad U^+_{\rm up} \cap U^+_{\rm down}. \tag{39}$$

Since the exterior derivative of (39) provides $d\omega^+_{\rm up} = d\omega^+_{\rm down}$ on $U^+_{\rm up} \cap U^+_{\rm down}$, the curvature form is globally defined by

$$\Omega^+ = \begin{cases} d\omega^+_{\rm up} & \text{on} \quad U^+_{\rm up}, \\ d\omega^+_{\rm down} & \text{on} \quad U^+_{\rm down}. \end{cases} \tag{40}$$

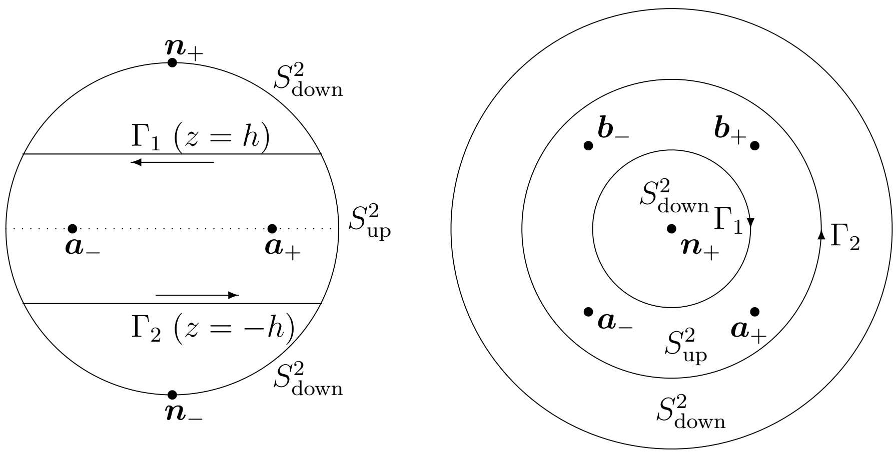

Figure 2: Division of the sphere into the disjoint union of $S^2_{\rm up}$ and $S^2_{\rm down}$ with $\omega^+_{\rm up}$ and $\omega^+_{\rm down}$ being smoothly defined on $S^2_{\rm up}$ and $S^2_{\rm down}$, respectively

In order to evaluate the Chern number c^+ of the eigen-line bundle associated with the positive eigenvalue λ^+,

$$c^+ = \frac{i}{2\pi}\int_{S^2} \Omega^+, \tag{41}$$

we draw two circles Γ_1 and Γ_2 determined, respectively, by $z = h$ and $z = -h$ with $0 < h < 1$ on the sphere S^2, and divide S^2 into three regions whose boundaries are these circles. Two of three regions containing the north and the south poles form a subset called $S^2_{\rm down}$ and the other one containing the equator is called $S^2_{\rm up}$, where we choose the orientation of each of the circles Γ_1 and Γ_2 so as to be in agreement with that of the $S^2_{\rm up}$. Since

$$S^2_{\rm up} \subset U^+_{\rm up}, \quad S^2_{\rm down} \subset U^+_{\rm down}, \tag{42}$$

and since the integral over S^2 is broken up into the sum of integrals over $S^2_{\rm up}$ and $S^2_{\rm down}$, the Stokes theorem is applied to evaluate the first Chern number as

$$c^+ = \frac{i}{2\pi}\int_{S^2}\Omega^+ = \frac{i}{2\pi}\Big(\int_{S^2_{\rm up}} d\omega^+_{\rm up} + \int_{S^2_{\rm down}} d\omega^+_{\rm down}\Big) = -\frac{1}{2\pi i}\int_{\Gamma_1+\Gamma_2}(\Phi^+)^{-1}d\Phi^+, \tag{43}$$

where use has been made of (39).

The right-hand side of (43) is minus the sum of the winding numbers of the maps $\Gamma_k \to U(1)$ by Φ^+ with $k=1,2$, which can be evaluated by using contour integrals along the circles Γ_1 and Γ_2. The curve Γ_1 is expressed as

$$x(t) = \sqrt{1-h^2}\cos t, \quad y(t) = \sqrt{1-h^2}\sin t, \quad z(t) = h. \tag{44}$$

We here denote the transition function by $\Phi^+ = \dfrac{X+iY}{\sqrt{X^2+Y^2}}$. In the case of $(a,b)=(1,-1)$, the functions X and Y along the curve Γ_1 are put in the form

$$\begin{aligned} X(t) &= \sqrt{3}(1-h^2)\cos 2t, \\ Y(t) &= \tfrac{1}{2}h(1-h^2)\sin 2t, \end{aligned} \tag{45}$$

where $(a,b)=(1,-1)$ has been taken instead of $(a,b)=(1,-1)/\sqrt{2}$ on account of the scale invariance. Since the orientation of Γ_1 is clockwise, the above equations show that the winding number is -2 for Γ_1. We can do the same reasoning to obtain the winding number -2 for Γ_2, where the expression of Γ_2 is given by (44) with $z=-h$ in place of $z=h$ and the orientation of Γ_2 is anti-clockwise. The sum of the winding numbers is -4. Hence, we have the Chern number $c^+=4$ for the parameters with $a>0, b<0$.

In the cases of other parameter values, the same method can be used to evaluate the Chern number. In the case of $a<0, b<0$, the definition of $U^+_{\rm up}$ and $U^+_{\rm down}$ becomes, apart from (36),

$$U^+_{\rm down} = S^2 - \{\boldsymbol{n}_\pm\}, \quad U^+_{\rm up} = S^2 - \{\boldsymbol{a}_\pm, \boldsymbol{b}_\pm\}. \tag{46}$$

We take two circles Γ_1 and Γ_2 in a similar manner to the case of $a>0$ to divide the sphere into three regions. In the present case, $S^2_{\rm up}$ is defined to be the union of the region in which either of $\boldsymbol{n}_\pm$ is contained, and $S^2_{\rm down}$ is the region containing the equator. The orientation of each of Γ_1 and Γ_2 is determined so as to be consistent with the orientation of $S^2_{\rm up}$. As a result, the orientations of Γ_1 and Γ_2 are opposite to those in the case of $a>0, b<0$. The formula (43) remains to hold true. Though the orientation of each curve is opposite, the functions (X,Y) receives the transformation $(X,Y)\to(-X,Y)$, so that the winding numbers are the same as those in the case of $a>0, b<0$. Further, the transformation $(a,b)\to(a,-b)$ of the parameters results in the change of Chern number in sign, as is easily verified. Thus, we have the Chern diagram given in Fig. 2.

Since the sum of Chern numbers of all the eigen-line bundles is zero, the Chern number c^- of the eigen-line bundle associated with the negative eigenvalue λ^- is minus the Chern number of the eigen-line bundle for the positive eigenvalue λ^+. This ends the proof.

5.2 Linearization at exceptional points

In what follows, we give another method to evaluate the Chern number. Instead of working with the contour integrals along Γ_1, Γ_2, we can deform the two curves into other four curves $\gamma_k, k = 1, \ldots, 4$, without changing the value of the initial contour integrals, where each of γ_k is a small circle centered at one of the exceptional points $\boldsymbol{a}_\pm, \boldsymbol{b}_\pm$. Hence, the Chern number is equal to minus the sum of winding numbers of γ_k;

$$c^+ = -\frac{1}{2\pi i}\int_{\Gamma_1+\Gamma_2} (\Phi^+)^{-1} d\Phi^+ = -\sum_k W(\gamma_k), \quad W(\gamma_k) = \frac{1}{2\pi i}\int_{\gamma_k} (\Phi^+)^{-1} d\Phi^+. \tag{47}$$

We here have to note that the orientation of γ_k is counter-clockwise, which results from the fact that the exceptional points $\boldsymbol{a}_\pm, \boldsymbol{b}_\pm$ are assigned to the "down" eigenvector in the case of $a > 0$. If we take up the exceptional points $\boldsymbol{n}_\pm$ assigned to the "up" eigenvector, the orientation of the small circles centered at respective exceptional points are clockwise. However, we don't take those exceptional points in the present case for the reason to be explained later.

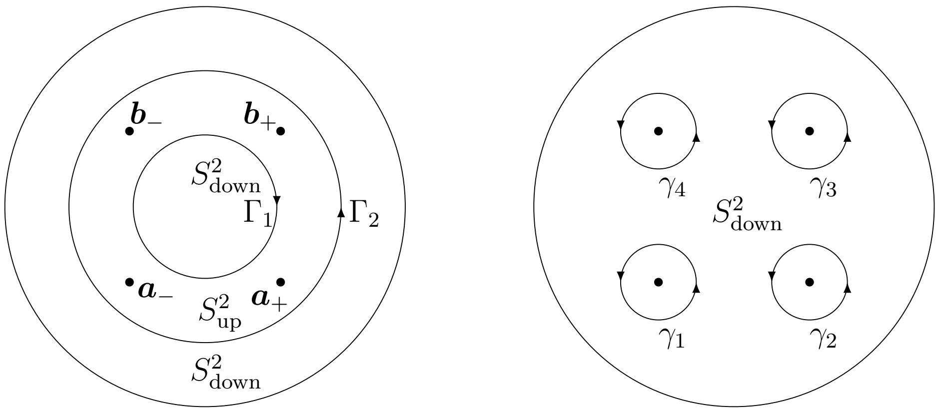

Figure 3: Deformation of contours $\Gamma_i, i = 1, 2$, into small circles $\gamma_j, j = 1, 2, 3, 4$, centered at exceptional points.

We do not need to explicitly calculate the contour integral along γ_k. The linearization of X and Y at each of the exceptional points is sufficient for us to know the winding number. For $\boldsymbol{a}_-$ and $\boldsymbol{b}_-$ with y-components negative, the local coordinates (x, z) are positively oriented, but for $\boldsymbol{a}_+$ and $\boldsymbol{b}_+$ with y-components positive, the local coordinates (z, x) are positively oriented. Since in the case of $a > 0, b < 0$, one has

$$\det \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial z} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial z} \end{pmatrix}_{\boldsymbol{a}_-,\boldsymbol{b}_-} < 0, \tag{48}$$

$$\det \begin{pmatrix} \frac{\partial X}{\partial z} & \frac{\partial X}{\partial x} \\ \frac{\partial Y}{\partial z} & \frac{\partial Y}{\partial x} \end{pmatrix}_{\boldsymbol{a}_+,\boldsymbol{b}_+} < 0, \tag{49}$$

the winding number assigned to each of $\boldsymbol{a}_\pm, \boldsymbol{b}_\pm$ is -1, so that the sum of them is -4. Hence the right-hand side of (47) or the Chern number is 4 in the case of $a > 0, b < 0$. This method was frequently used in [4] (see Eq.(71) in [4]).

In concluding this section, we have to note that the linearization method adopted above can not be applied to small circles centered at $\boldsymbol{n}_\pm$. This is because the corresponding determinant evaluated at each of $\boldsymbol{n}_\pm$ vanishes. However, if we take another basis for expressing the Hamiltonian, the exceptional points appear in another form and this discrepancy disappears (see Sec. 11.3).

6 Linearization and retraction to a two-level system

In order to work with Chern numbers of eigen-line bundles, we need to know eigenvalues and eigenvectors by solving the eigenvalue equation, if we follow the standard method. However, characteristic equations of degree greater than two are not easy to solve. If we can break up the problem into parts which are tractable, we can reach a solution.

The condition for degeneracy in eigenvalues for Hermitian matrices defined on the total parameter space $\mathbb{R}^m \times S^2$ splits the control parameter space $\mathbb{R}^m$ into disjoint domains in general. Since degeneracy in eigenvalues occurs for any point of the boundaries of such domains, we refer to the boundaries of domains as degeneracy surfaces (or curves, if one-dimensional) in the control parameter space. We are then interested in transition states which emerges when the parameter value varies from a domain into another across a degeneracy surface in the control parameter space. We hope that though the Chern number itself is not easy to evaluate for regular values of the parameter a change or modification in Chern number may be evaluated, which accompanies a change in parameter values in crossing a boundary surface in the control parameter space.

With this in mind, we look into what happens in eigen-line bundles at a degeneracy point. For simplicity, we treat a three-level model, *i.e.*, a traceless 3×3 Hermitian matrix as a full Hamiltonian. For a regular value of the parameter, the eigenvalues of the Hamiltonian $H(\boldsymbol{x})$ are distinct, which we denote by $\lambda_1(\boldsymbol{x}) > \lambda_2(\boldsymbol{x}) > \lambda_3(\boldsymbol{x})$. Let $|e_k(\boldsymbol{x})\rangle$ be the normalized eigenvector associated with the $\lambda_k(\boldsymbol{x})$. Then, the whole space $\mathbb{C}^3$ is decomposed, for any point $\boldsymbol{x} \in S^2$, into

$$\mathbb{C}^3 = \mathrm{span}\{|e_1(\boldsymbol{x})\rangle\} \oplus \mathrm{span}\{|e_2(\boldsymbol{x})\rangle\} \oplus \mathrm{span}\{|e_3(\boldsymbol{x})\rangle\}. \tag{50}$$

Assigning each eigenspace $\mathrm{span}\{|e_k(\boldsymbol{x})\rangle\}$ to $\boldsymbol{x} \in S^2$, one can form an eigen-line bundle $L_k \to S^2$, so that the trivial bundle $\mathbb{C}^3 \times S^2$ breaks up into the direct sum $L_1 \oplus L_2 \oplus L_3$. If the parameter takes a singular value sitting on a degeneracy surface, a degeneracy occurs in the eigenvalues, which take the form, for example, $\lambda_1(\boldsymbol{x}_0) > \lambda_2(\boldsymbol{x}_0) = \lambda_3(\boldsymbol{x}_0)$ at a certain point $\boldsymbol{x}_0 \in S^2$. Then the corresponding decomposition of $\mathbb{C}^3$ becomes, at $\boldsymbol{x}_0$,

$$\mathbb{C}^3 = \mathrm{span}\{|e_1(\boldsymbol{x}_0)\rangle\} \oplus \mathrm{span}\{|e_2(\boldsymbol{x}_0)\rangle, |e_3(\boldsymbol{x}_0)\rangle\}. \tag{51}$$

At this moment, the direct sum $L_1 \oplus L_2 \oplus L_3$ breaks down. However, if the parameter value changes into a regular value, then we have the decomposition of the form (50) throughout S^2, but the corresponding eigen-line bundles do not remain to be the same, as is easily seen from examples of Chern diagrams (see Fig. 1). By $L'_k \to S^2$, $k = 1, 2, 3$, we denote the reconstructed eigen-line bundles. The transition from L_k to L'_k corresponds to the reorganization of band structure.

A question arises as to whether one can use the subspace $\mathrm{span}\{|e_2(\boldsymbol{x}_0)\rangle, |e_3(\boldsymbol{x}_0)\rangle\}$ with $\boldsymbol{x}_0$ being isolated, in order to evaluate a change in Chern numbers against the variation in control parameters by means of the linear approximation of the Hamiltonian on this subspace or not. If the answer is affirmative, a further question arises as to whether this method can be extended to apply to n-level system with $n \geq 3$. However, if only a pair of eigenvalues are degenerate at once for a set of critical parameter values, three-level systems are enough for us to observe how the bundle structure changes against parameters. This is because, the doubly degenerate eigenvalues are responsible for the topological change but the other non-degenerate eigenvalues are robust and do not contribute to the topological change. In what follows, we restrict ourselves to 3×3 Hermitian matrices (or 3-level model Hamiltonians).

6.1 Local Hamiltonians

Let $c(t)$ be a curve transverse to a degeneracy surface in the control parameter space. We assume that the Hamiltonian has doubly degenerate eigenvalues at an isolated point $\boldsymbol{x}_0 \in S^2$ for $t = 0$ only in the vicinity of $t = 0$. For Hamiltonians with symmetry by a finite subgroup G of $SO(3)$, the set of degeneracy points forms an orbit of G in general. We put the Hamiltonian in the form $H(c(t), \boldsymbol{x})$ to explicitly indicate the parameter dependence. Let $\boldsymbol{\xi}_k$, $k = 1, 2$, be a frame or a positively-oriented orthonormal system of tangent vectors to S^2 at $\boldsymbol{x}_0$. The tangent plane Π_0 to S^2 at $\boldsymbol{x}_0$ is endowed with the Cartesian coordinates through $\sum q_k \boldsymbol{\xi}_k$. Now, the local Hamiltonian is defined to be

$$H_{\mathrm{loc}}(t, q; \boldsymbol{x}_0) = H(c(0), \boldsymbol{x}_0) + t\dot{H}(c(0), \boldsymbol{x}_0) + \sum q_k \nabla H(c(0), \boldsymbol{x}_0) \cdot \boldsymbol{\xi}_k, \tag{52}$$

where the $\dot{H}$ denotes the derivative of H with respect to t and ∇ the gradient operator with respect to $\boldsymbol{x}$, and where $\nabla H \cdot \boldsymbol{\xi}_k$ is the matrix whose (j, ℓ)-components are $\nabla H_{j\ell} \cdot \boldsymbol{\xi}_k$ with the dot $\cdot$ denoting the inner product already defined (see Eq. (7)). We note that $H_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ is a traceless Hermitian matrix as well as $H(c(t), \boldsymbol{x})$.

Since $H(c(t), \boldsymbol{x})$ admits the G symmetry, the local Hamiltonian $H_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ also admits the symmetry by the isotropy subgroup G_0 at $\boldsymbol{x}_0$: For $h \in G_0$, the $H_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ is subject to the transformation

$$\mathrm{Ad}_{D(h)} H_{\mathrm{loc}}(t, q; \boldsymbol{x}_0) = H_{\mathrm{loc}}(t, h^{(2)} q; \boldsymbol{x}_0), \quad h \in G_0, \tag{53}$$

as is easily seen from (11), where $D(h)$ is a representation matrix for G_0 and where $h^{(2)}$ denotes the matrix acting on Π_0, which is defined through $h\boldsymbol{\xi}_k = \sum_j h^{(2)}_{jk} \boldsymbol{\xi}_j$ with respect to the frame $\boldsymbol{\xi}_k$ at $\boldsymbol{x}_0$.

As is easily observed from (8), for $g \in G$, the local Hamiltonian transforms according to

$$\mathrm{Ad}_{D(g)} H_{\mathrm{loc}}(t, q; \boldsymbol{x}_0) = H_{\mathrm{loc}}(t, q; g\boldsymbol{x}_0), \tag{54}$$

if the frame $g\boldsymbol{\xi}_k$ is adopted at $g\boldsymbol{x}_0$,

If $g_1 \boldsymbol{x}_0 = g\boldsymbol{x}_0$, we have the transformation

$$\mathrm{Ad}_{D(g_1)} H_{\mathrm{loc}}(t, q; \boldsymbol{x}_0) = H_{\mathrm{loc}}(t, h^{(2)} q, g_1 \boldsymbol{x}_0), \tag{55}$$

which results from (10) with $h^{(2)}$ being defined in (9), where $g_1 = hg$ with h being an element of the isotropy subgroup at $g\boldsymbol{x}_0$.

6.2 Local Hamiltonians in normal form

We assume that $\lambda_1(\boldsymbol{x}_0) > \lambda_2(\boldsymbol{x}_0) = \lambda_3(\boldsymbol{x}_0)$ for $t = 0$. Let $|e_1(\boldsymbol{x}_0)\rangle$ be a normalized eigenvector associated with $\lambda_1(\boldsymbol{x}_0)$ and $|e_2(\boldsymbol{x}_0)\rangle, |e_3(\boldsymbol{x}_0)\rangle$ orthonormal eigenvectors associated with $\lambda_2(\boldsymbol{x}_0) = \lambda_3(\boldsymbol{x}_0)$, where $|e_1(\boldsymbol{x}_0)\rangle$ and $\{|e_2(\boldsymbol{x}_0)\rangle, |e_3(\boldsymbol{x}_0)\rangle\}$ are determined up to $U(1)$ and $U(2)$, respectively.

Instead of using the standard basis of $\mathbb{C}^3$, we may choose $|e_k(\boldsymbol{x}_0)\rangle$ as a basis with respect to which the local Hamiltonian $H_{\rm loc}(t, q; \boldsymbol{x}_0)$ is expressed. Owing to the symmetry, the isotropy subgroup G_0 leaves each eigenspace invariant. Since the G_0 is isomorphic to a cyclic group and since it is Abelian, the representation of G_0 on $\mathrm{span}\{|e_2(\boldsymbol{x}_0)\rangle, |e_3(\boldsymbol{x}_0)\rangle\}$ is reducible to a direct sum of two one-dimensional vector subspaces. Each of these subspaces carries an irreducible representation of G_0. We then may take a basis $|e'_k(\boldsymbol{x}_0)\rangle$ of $\mathbb{C}^3$ with respect to which G_0 is represented in the form of a diagonal matrix. We denote by $K_{\rm loc}(t, q; \boldsymbol{x}_0)$ the local Hamiltonian which is expressed with respect to the basis thus chosen, and refer to $K_{\rm loc}(t, q; \boldsymbol{x}_0)$ as the local Hamiltonian in normal form. Let $U_3 = (|e'_1(\boldsymbol{x}_0)\rangle, |e'_2(\boldsymbol{x}_0)\rangle, |e'_3(\boldsymbol{x}_0)\rangle)$ be the unitary matrix formed by the new basis. Since $K_{\rm loc}(t, q; \boldsymbol{x}_0) = U_3^{-1} H_{\rm loc}(t, q; \boldsymbol{x}_0) U_3$, all elements of $K_{\rm loc}(t, q; \boldsymbol{x}_0)$ as well as those of $H_{\rm loc}(t, q; \boldsymbol{x}_0)$ are linear in t, q_1, q_2. In particular, since $K_{\rm loc}(0, 0; \boldsymbol{x}_0)$ is of diagonal form, the off-diagonal elements $\kappa_{jk}(t, q)$ of $K_{\rm loc}(t, q; \boldsymbol{x}_0)$ with $j \neq k$ must be homogeneously linear in t, q_1, q_2.

Like (53), the local Hamiltonian $K_{\rm loc}(t, q; \boldsymbol{x}_0)$ transforms according to

$$\mathrm{Ad}_{\tilde{D}(h)} K_{\rm loc}(t, q; \boldsymbol{x}_0) = K_{\rm loc}(t, h^{(2)} q; \boldsymbol{x}_0), \quad h \in G_0, \tag{56}$$

where $\tilde{D}(h)$ denotes the representation matrix expressed with respect to the basis $|e'_k(\boldsymbol{x}_0)\rangle$ mentioned above, taking a diagonal form. It is of great help to give here an example in order to understand what the symmetry condition (56) implies. We assume that $G_0 \cong C_3$ and $\tilde{D}(h)$ takes the form

$$\tilde{D}(h) = \begin{pmatrix} e^{2\pi i/3} & & \\ & e^{-2\pi i/3} & \\ & & 1 \end{pmatrix} \tag{57}$$

and further that $h^{(2)}$ is expressed with respect to a suitably chosen frame $\boldsymbol{\xi}_k$ as

$$h^{(2)} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}. \tag{58}$$

With this setting, Eq. (56) is written out as

$$\begin{aligned} &\begin{pmatrix} \kappa_{11}(t,q) & e^{-2\pi i/3}\kappa_{12}(t,q) & e^{2\pi i/3}\kappa_{13}(t,q) \\ e^{2\pi i/3}\kappa_{21}(t,q) & \kappa_{22}(t,q) & e^{-2\pi i/3}\kappa_{23}(t,q) \\ e^{-2\pi i/3}\kappa_{31}(t,q) & e^{2\pi i/3}\kappa_{32}(t,q) & \kappa_{33}(t,q) \end{pmatrix} \\ &= \begin{pmatrix} \kappa_{11}(t,h^{(2)}q) & \kappa_{12}(t,h^{(2)}q) & \kappa_{13}(t,h^{(2)}q) \\ \kappa_{21}(t,h^{(2)}q) & \kappa_{22}(t,h^{(2)}q) & \kappa_{23}(t,h^{(2)}q) \\ \kappa_{31}(t,h^{(2)}q) & \kappa_{32}(t,h^{(2)}q) & \kappa_{33}(t,h^{(2)}q) \end{pmatrix}, \end{aligned} \tag{59}$$

which implies that the diagonal elements $\kappa_{kk}(t,q)$ are independent of q and the off-diagonal elements $\kappa_{jk}(t,q)$ with $k \neq j$ are independent of t and further that $\kappa_{12}(t,q)$ and $\kappa_{23}(t,q)$ are proportional to $q_1 - iq_2$ and $\kappa_{13}(t,q)$ is proportional to $q_1 + iq_2$ (see (182) as a realization of (59)).

For $g, g_1 \in G$, we have the same transformation rules as in (54) and (55),

$$\mathrm{Ad}_{\tilde{D}(g)} K_{\mathrm{loc}}(t,q;\boldsymbol{x}_0) = K_{\mathrm{loc}}(t,q;g\boldsymbol{x}_0), \tag{60}$$

$$\mathrm{Ad}_{\tilde{D}(g_1)} K_{\mathrm{loc}}(t,q;\boldsymbol{x}_0) = K_{\mathrm{loc}}(t,h^{(2)}q,g_1\boldsymbol{x}_0), \tag{61}$$

respectively.

6.3 Approximation of eigenvalues for sufficiently small t, q

Let $K_{\mathrm{loc}}(t,q;\boldsymbol{x}_0)$ be a local Hamiltonian in normal form. In view of the decomposition (51), we put $K_{\mathrm{loc}}(t,q;\boldsymbol{x}_0)$ in the form

$$K_{\mathrm{loc}}(t,q;\boldsymbol{x}_0) = \begin{pmatrix} a_{00} & c_{01} & c_{02} \\ \overline{c}_{01} & a_{11} & c_{12} \\ \overline{c}_{02} & \overline{c}_{12} & a_{22} \end{pmatrix}, \tag{62}$$

and pick up the lower right 2×2 block matrix by setting

$$K^{(2)}_{\mathrm{loc}}(t,q;\boldsymbol{x}_0) = \begin{pmatrix} a_{11} & c_{12} \\ \overline{c}_{12} & a_{22} \end{pmatrix}, \tag{63}$$

where all off-diagonal elements are homogeneously linear in t, q_1, q_2.

The eigenvalues $\lambda^{\pm}$ of $K^{(2)}_{\mathrm{loc}}(t,q;\boldsymbol{x}_0)$ are given by (21) with a_{11}, a_{22}, c_{12} being defined on the tangent plane Π_0 at $\boldsymbol{x}_0$. For each of the eigenvalues $\lambda^{\pm}$, we have two expressions of the normalized associated eigenvectors;

$$|u^{\pm}_{\mathrm{up}}(t,q)\rangle = \frac{1}{\sqrt{(\lambda^{\pm} - a_{11})^2 + |c_{12}|^2}} \begin{pmatrix} c_{12} \\ \lambda^{\pm} - a_{11} \end{pmatrix}, \quad q \in U^{\pm}_{\mathrm{up}}, \tag{64}$$

$$|u^{\pm}_{\mathrm{down}}(t,q)\rangle = \frac{1}{\sqrt{(\lambda^{\pm} - a_{22})^2 + |c_{12}|^2}} \begin{pmatrix} \lambda^{\pm} - a_{22} \\ \overline{c}_{12} \end{pmatrix}, \quad q \in U^{\pm}_{\mathrm{down}}. \tag{65}$$

Here, points at which the quantities under the square root symbols vanish are called exceptional points and the domains $U^{\pm}_{\mathrm{up/down}}$ of respective eigenvectors are the tangent plane with the assigned exceptional points removed.

We wish to show that two of the eigenvalues of $K_{\mathrm{loc}}(t,q;\boldsymbol{x}_0)$ can be approximated by the eigenvalues of $K^{(2)}_{\mathrm{loc}}(t,q;\boldsymbol{x}_0)$ if t, q_1, q_2 are sufficiently small. Let $F(\mu)$ denote the characteristic polynomial for $K_{\mathrm{loc}}(t,q;\boldsymbol{x}_0)$. Let $\lambda^{\pm}$ be the eigenvalues of $K^{(2)}_{\mathrm{loc}}(t,q;\boldsymbol{x}_0)$. Then, for $\mu = \lambda^{\pm}$, $F(\mu)$ takes the values

$$F(\lambda^{\pm}) = -c_{01}(\overline{c}_{01}(a_{22} - \lambda^{\pm}) - c_{12}\overline{c}_{02}) + c_{02}(\overline{c}_{01}\overline{c}_{12} - c_{02}(a_{11} - \lambda^{\pm})). \tag{66}$$

We here note that $a_{11} = a_{22}$ for $(t, q) = (0, 0)$, which are equal to the degenerate eigenvalues $\lambda_2(\boldsymbol{x}_0) = \lambda_3(\boldsymbol{x}_0)$, so that $a_{11} - a_{22}$ is homogeneously linear in t, q_1, q_2. On this account together with the fact that c_{12} is homogeneous linear in t, q_1, q_2, both

$$\lambda^{\pm} - a_{11} = \frac{1}{2}\Big(a_{22} - a_{11} \pm \sqrt{(a_{11} - a_{22})^2 + 4|c_{12}|^2}\Big), \tag{67a}$$

$$\lambda^{\pm} - a_{22} = \frac{1}{2}\Big(a_{11} - a_{22} \pm \sqrt{(a_{11} - a_{22})^2 + 4|c_{12}|^2}\Big) \tag{67b}$$

are of the first order in t, q_1, q_2. Further, c_{01}, c_{02} are homogeneously linear in t, q_1, q_2 as well. Hence, the right-hand side of (66) is of the third order in t, q_1, q_2, which implies that $\lambda^{\pm}$ well approximate to eigenvalues of $K_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ if t, q_1, q_2 are sufficiently small.

We turn to the eigenvectors associated with the eigenvalues approximate to $\lambda^{\pm}$. Let μ be an eigenvalue of $K_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$. From the first and the second equations of the eigenvalue equations composed of three, we obtain an "up" eigenvector

$$\begin{pmatrix} -c_{01}c_{12} + (a_{11} - \mu)c_{02} \\ (a_{00} - \mu)c_{12} - \bar{c}_{01}c_{02} \\ |c_{01}|^2 - (a_{00} - \mu)(a_{11} - \mu) \end{pmatrix}_{\mathrm{up}}. \tag{68}$$

If we replace μ in (68) by the eigenvalues $\lambda^{\pm}$ of $K^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ and if we pick up the terms of order less than two in t, q_1, q_2 from the above vector with $\lambda^{\pm}$ for μ, we obtain the vector

$$[a_{00} - \lambda^{\pm}]_0 \begin{pmatrix} 0 \\ c_{12} \\ \lambda^{\pm} - a_{11} \end{pmatrix}_{\mathrm{up}}, \tag{69}$$

where $[a_{00} - \lambda^{\pm}]_0$ denotes the constant term from $a_{00} - \lambda^{\pm}$. The obtained vector is reminiscent of $|u^{\pm}_{\mathrm{up}}(q)\rangle$ given in (64), if normalized. Put another way, for sufficiently small t, q_1, q_2, the normalized "up" eigenvectors associated with the eigenvalues approximate to $\lambda^{\pm}$ project to $|u^{\pm}_{\mathrm{up}}(q)\rangle$.

From the first and the third equations of the eigenvalue equations for $K_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$, we obtain a "down" eigenvector

$$\begin{pmatrix} c_{01}(a_{22} - \mu) - c_{02}\bar{c}_{12} \\ |c_{02}|^2 - (a_{00} - \mu)(a_{22} - \mu) \\ \bar{c}_{12}(a_{00} - \mu) - c_{01}\bar{c}_{02} \end{pmatrix}_{\mathrm{down}}. \tag{70}$$

In a similar manner to the above, we obtain the vector

$$[a_{00} - \lambda^{\pm}]_0 \begin{pmatrix} 0 \\ \lambda^{\pm} - a_{22} \\ \bar{c}_{12} \end{pmatrix}_{\mathrm{down}}, \tag{71}$$

where $[a_{00} - \lambda^{\pm}]_0$ denotes the constant term from $a_{00} - \lambda^{\pm}$. The resultant vector is reminiscent of $|u^{\pm}_{\mathrm{down}}(q)\rangle$ given by (65), if normalized. Thus, if t, q_1, q_2 are sufficiently small, the normalized "down" eigenvectors associated with the eigenvalues approximate to $\lambda^{\pm}$ project to $|u^{\pm}_{\mathrm{down}}(q)\rangle$.

6.4 Retraction to a two-level system

So far we have shown that two of the eigenvalues and the associated eigenvectors for $K_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ are approximated by those for $K^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$. We now consider the relation between eigen-line bundles for $K_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ and those for $K^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$.

We denote the normalized eigenvectors of $K_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ by $|v^{\pm}_{\mathrm{up}}(t, q)\rangle$ and $|v^{\pm}_{\mathrm{down}}(t, q)\rangle$, where the superscripts $\pm$ are attached to the eigenvectors in association with eigenvalues close to $\lambda^{\pm}$, respectively. The corresponding transition function $\Phi^{\pm}_{(3)}(t, q)$ is defined through

$$|v^{\pm}_{\mathrm{up}}(t, q)\rangle = \Phi^{\pm}_{(3)}(t, q)|v^{\pm}_{\mathrm{down}}(t, q)\rangle, \quad q \in V^{\pm}_{\mathrm{up}} \cap V^{\pm}_{\mathrm{down}}, \tag{72}$$

where $V^{\pm}_{\mathrm{up}}$ and $V^{\pm}_{\mathrm{down}}$ denote the domains of $|v^{\pm}_{\mathrm{up}}(t, q)\rangle$ and $|v^{\pm}_{\mathrm{down}}(t, q)\rangle$, respectively.

In comparison with this, the transition function $\Phi^{\pm}_{(2)}(t, q)$ between eigenvectors $|u^{\pm}_{\mathrm{up}}(t, q)\rangle$ and $|u^{\pm}_{\mathrm{down}}(t, q)\rangle$ is defined through

$$|u^{\pm}_{\mathrm{up}}(t, q)\rangle = \Phi^{\pm}_{(2)}(t, q)|u^{\pm}_{\mathrm{down}}(t, q)\rangle, \quad q \in U^{\pm}_{\mathrm{up}} \cap U^{\pm}_{\mathrm{down}}. \tag{73}$$

Since for sufficiently small t, q_1, q_2, $|v^{\pm}_{\mathrm{up}}(t, q)\rangle$ and $|v^{\pm}_{\mathrm{down}}(t, q)\rangle$, if projected, can be approximated by $|u^{\pm}_{\mathrm{up}}(t, q)\rangle$ and $|u^{\pm}_{\mathrm{down}}(t, q)\rangle$, respectively, the transition functions $\Phi^{\pm}_{(3)}(t, q)$ can also be approximated by $\Phi^{\pm}_{(2)}(t, q)$, respectively, if t, q_1, q_2 are small enough. It then turns out that the eigen-line bundles for the Hamiltonian $K^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ approximate to the corresponding two of those for the Hamiltonian $K_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ in the sense that the transition functions $\Phi^{\pm}_{(2)}(t, q)$ approximate to $\Phi^{\pm}_{(3)}(t, q)$.

7 The two-level system at a degeneracy point

So far we have observed that as long as transition functions of eigen-line bundles are concerned, the local Hamiltonian $K_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ may be replaced by $K^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$. We now concentrate on the Hamiltonian $K^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ from the viewpoint of Chern number modification accompanying the crossing of a boundary of an iso-Chern domain.

7.1 Geometric setting on the traceless 2×2 Hermitian matrices

The condition for a 2×2 Hermitian matrix $H^{(2)}$ to have degenerate eigenvalues is given by $\det H^{(2)} - \frac{1}{4}(\mathrm{tr}\, H^{(2)})^2 = 0$ or equivalently by $\lambda_1\lambda_2 - \frac{1}{4}(\lambda_1 + \lambda_2)^2 = 0$ for the eigenvalues λ_1, λ_2 of $H^{(2)}$. Since

$$\det\bigl(H^{(2)} - \frac{1}{2}(\mathrm{tr}\, H^{(2)})I\bigr) = \det H^{(2)} - \frac{1}{4}(\mathrm{tr}\, H^{(2)})^2, \tag{74}$$

the eigenvalues of $H^{(2)}$ are degenerate if and only if $\det\bigl(H^{(2)} - \frac{1}{2}(\mathrm{tr}\, H^{(2)})I\bigr) = 0$. The latter condition is equivalent to the condition for the traceless Hermitian matrix $H^{(2)} - \frac{1}{2}(\mathrm{tr}\, H^{(2)})I$ to have degenerate eigenvalues. Further, as is seen from (27), the transition function is independent of diagonal elements except for the factors $\varepsilon^{\pm}$, but the $\varepsilon^{\pm}$ are irrelevant to the winding number and hence to the Chern number. Thus, as long as we are interested in degeneracy of eigenvalues of $H^{(2)}$ and in related eigen-line bundles, we are allowed to study the traceless matrix $H^{(2)} - \frac{1}{2}(\operatorname{tr} H^{(2)})I$ in place of $H^{(2)}$. For this reason, our interest centers on the traceless 2×2 Hermitian matrices. We have already denoted by $\mathcal{H}_0(2)$ the set of the traceless 2×2 Hermitian matrices.

On introducing the Pauli matrices with suffices shifted from the usual ones,

$$\sigma'_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad \sigma'_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma'_3 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \tag{75}$$

the $\mathcal{H}_0(2)$ is shown to be isomorphic with $\mathbb{R}^3$ through the isomorphism defined by

$$\mathcal{H}_0(2) \to \mathbb{R}^3; \quad \sum_k p_k \sigma'_k \mapsto \boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}. \tag{76}$$

The $\mathcal{H}_0(2)$ is endowed with a metric through

$$-\det(\sum_k p_k \sigma'_k) = \sum_k p_k^2, \tag{77}$$

which is isometric with the standard metric on $\mathbb{R}^3$.

We now show that the adjoint action of a unitary matrix on $\mathcal{H}_0(2)$ gives rise to an orthogonal transformation on $\mathbb{R}^3$. Let U be a 2×2 unitary matrix, which acts on $\mathcal{H}_0(2)$ by adjoint. Since the adjoint action on the basis σ'_k is described as

$$U \sigma'_k U^{-1} = \sum_j h_{jk} \sigma'_j, \quad k = 1, 2, 3, \tag{78}$$

we have

$$U \sum_k p_k \sigma'_k U^{-1} = \sum_{k,j} h_{jk} p_k \sigma'_j, \tag{79}$$

and hence

$$\det\Big(U \sum_k p_k \sigma'_k U^{-1}\Big) = \det\Big(\sum_k p_k \sigma'_k\Big), \tag{80}$$

which implies that $\|h\boldsymbol{p}\| = \|\boldsymbol{p}\|$ with $h = (h_{jk})$. Thus, the adjoint action on $\mathcal{H}_0(2)$ by $U \in U(2)$ induces the orthogonal transformation $\boldsymbol{p} \mapsto h\boldsymbol{p}$ on $\mathbb{R}^3$ by $h \in O(3)$. Since the identity of $U(2)$ corresponds to the identity of $O(3)$ and since $U(2)$ is connected, the target space $O(3)$ must be restricted to $SO(3)$. Thus we have the map

$$T : U(2) \to SO(3); \quad U \mapsto h = (h_{jk}). \tag{81}$$

The kernel of this map is of course isomorphic to $U(1)$.

7.2 Relevant 3×3 real matrices

Let $K^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ be the retract from a local Hamiltonian $K_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ in normal form. As is mentioned in Sec. 6.4, from the viewpoint of associated eigen-line bundles, we may make $K^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ into the traceless Hamiltonian

$$\tilde{K}^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0) := K^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0) - \frac{1}{2}(\operatorname{tr} K^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0))I. \tag{82}$$

Since the diagonal elements of $K_{\rm loc}^{(2)}(t,q;\boldsymbol{x}_0)$ are degenerate eigenvalues for $(t,q)=(0,0)$, the diagonal elements of $\tilde{K}_{\rm loc}^{(2)}(t,q;\boldsymbol{x}_0)$ become homogeneously linear in t,q_1,q_2 as well as the off-diagonal elements. For notational simplicity, we set $K(t,q)=\tilde{K}_{\rm loc}^{(2)}(t,q;\boldsymbol{x}_0)$ and put K in the form

$$K(t,q)=\begin{pmatrix} K_{11} & K_{12} \\ \overline{K}_{12} & -K_{11} \end{pmatrix}, \tag{83}$$

where K_{11} and K_{12} are homogeneously linear in $(t,q_1,q_2)\in\mathbb{R}^3$. Taking the basis (75) into account, we express them as

$$K_{11} = a_{11}t+b_{11}q_1+c_{11}q_2, \tag{84a}$$
$$K_{12} = a_{12}t+b_{12}q_1+c_{12}q_2-i(a_{13}t+b_{13}q_1+c_{13}q_2), \tag{84b}$$

where all coefficients, $a_{11},\ldots,c_{13}$ are real, and the present a_{11} and c_{12} are different from those used in (20). From these coefficients, we introduce the following real vectors

$$\boldsymbol{a}=\begin{pmatrix} a_{11}\\ a_{12}\\ a_{13}\end{pmatrix},\ \boldsymbol{b}=\begin{pmatrix} b_{11}\\ b_{12}\\ b_{13}\end{pmatrix},\ \boldsymbol{c}=\begin{pmatrix} c_{11}\\ c_{12}\\ c_{13}\end{pmatrix}. \tag{85}$$

According to the isomorphism $\mathcal{H}_0(2)\to\mathbb{R}^3$, we have the correspondence

$$K(t,q)\mapsto \boldsymbol{p}(t,q)=\boldsymbol{a}t+\boldsymbol{b}q_1+\boldsymbol{c}q_2. \tag{86}$$

Putting together these vectors, we define the 3×3 real matrix by

$$C(K(t,q))=(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}). \tag{87}$$

Then, the right-hand side of (86) takes the form

$$\boldsymbol{p}(t,q)=C(K(t,q))\begin{pmatrix} t\\ q\end{pmatrix}, \tag{88}$$

where $q=\binom{q_1}{q_2}\in\mathbb{R}^2$.

The vector space $\mathbb{R}\times\mathbb{R}^2$ to which $\begin{pmatrix} t\\ q\end{pmatrix}$ belongs is viewed as a tangent space to $\Gamma\times S^2$ at $(c(0),\boldsymbol{x}_0)$, where Γ denotes the curve $c(t)$ passing a degeneracy point in the control parameter space when $t=0$. If we denote by Λ_0 the tangent line to the curve Γ at $c(0)$ and by Π_0 the tangent plane to S^2 at the degeneracy point $\boldsymbol{x}_0\in S^2$, the vector space to which $\begin{pmatrix} t\\ q\end{pmatrix}$ belongs is described as $\Lambda_0\times\Pi_0$.

7.3 Left and right actions of $SO(3)\times SO(2)$ on $C(\tilde{K}_{\rm loc}^{(2)}(t,q;\boldsymbol{x}_0))$

The adjoint action by $U\in U(2)$ on $K(t,q)$ induces the left action of $T(U)=h\in SO(3)$ on $\boldsymbol{p}(t,q)$, which gives rise to the left action of h on $C=C(K(t,q))$;

$$C\mapsto hC=(h\boldsymbol{a},h\boldsymbol{b},h\boldsymbol{c}),\quad h\in SO(3), \tag{89}$$

as is easy seen form (88).

We here recall that the rotation group $SO(2)$ acts on the tangent plane Π_0 through $\sum \kappa^{(2)}_{jk}\boldsymbol{\xi}_j$with $\kappa^{(2)} = (\kappa^{(2)}_{jk}) \in SO(2)$, which means a change of frames on the tangent plane. This $SO(2)$ action is expressed as $\sum_k \kappa^{(2)}_{jk} q_k$ in the coordinates. In view of (88), we see that the rotation matrix $\kappa^{(2)}$ gives rise to the right action on the matrix C through

$$C \mapsto C\kappa^{(3)}, \quad \kappa^{(3)} =: \begin{pmatrix} 1 & \\ & \kappa^{(2)} \end{pmatrix} \in SO(2) \subset SO(3). \tag{90}$$

Thus we have found that $SO(3) \times SO(2)$ acts on $C(K(t, q))$ in the manner

$$C(K(t, q)) \mapsto hC(K(t, q))\kappa^{(3)}, \quad (h, \kappa^{(3)}) \in SO(3) \times SO(2). \tag{91}$$

As is easily seen from (91), $\det C(K(t, q))$ is invariant under the left and right action of $SO(3) \times SO(2)$. This implies that $\det C(K(t, q))$ is independent of the choice of the basis with respect to which the retracted local Hamiltonian $\tilde{K}^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ with vanishing trace is expressed, and of the choice of the frame on the tangent plane Π_0, as long as the frame in question is positively oriented.

7.4 Invariance of $\det C(\tilde{K}^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0))$ on the G-orbit

So far we have studied the local Hamiltonian at a degeneracy point $\boldsymbol{x}_0$. As the set of degeneracy points on S^2 forms an orbit of the symmetry group G, we are interested in the dependence of $\det\big(C(\tilde{K}^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0))\big)$ on degeneracy points. Let $\boldsymbol{x}'_0 = g\boldsymbol{x}_0$, $g \in G$ be another degeneracy point. Let $|e_j(\boldsymbol{x}_0)\rangle$ be a basis with respect to which the local Hamiltonian $K_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ is in normal form. The vectors $|e_j(\boldsymbol{x}'_0)\rangle := \tilde{D}(g)|e_j(\boldsymbol{x}_0)\rangle$ are orthonormal eigenvectors of $H(c(0), \boldsymbol{x}'_0)$. If we take the basis $|e_j(\boldsymbol{x}'_0)\rangle$ instead of $|e_j(\boldsymbol{x}_0)\rangle$, and adopt the frame $\boldsymbol{\xi}'_k = g\boldsymbol{\xi}_k$ on the tangent plane Π'_0 at $\boldsymbol{x}'_0$, then the local Hamiltonian $K'_{\mathrm{loc}}(t, q; \boldsymbol{x}'_0)$ on Π'_0 takes the same form as the local Hamiltonian on Π_0,

$$K'_{\mathrm{loc}}(t, q; \boldsymbol{x}'_0) = K_{\mathrm{loc}}(t, q; \boldsymbol{x}_0), \tag{92}$$

which results from (60). Hence, retracting the local Hamiltonian to the eigenspace associated with the degenerate eigenvalues, we have

$$\tilde{K}'^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}'_0) = \tilde{K}^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0). \tag{93}$$

If a frame other than $\boldsymbol{\xi}'_k$ is adopted, the left-hand side of the above equation becomes of the form $\tilde{K}'^{(2)}_{\mathrm{loc}}(t, \kappa^{(2)}q; \boldsymbol{x}'_0)$ with $\kappa^{(2)} \in SO(2)$. As is already shown in the preceding subsection, $\det\big(C(\tilde{K}'^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}'_0))\big)$ is independent of the choice of the basis $|e_k(\boldsymbol{x}'_0)\rangle$ and of the choice of the frame $\boldsymbol{\xi}'_k$, we find that $\det C(\tilde{K}'^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}'_0))$ is independent of the choice of a degeneracy point $\boldsymbol{x}'_0$ in the orbit of $\boldsymbol{x}_0$ by G. Put another way, $\det\big(C(\tilde{K}^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0))\big)$ is invariant on the orbit $\mathcal{O}_{\boldsymbol{x}_0}$ of the symmetry group G.

Proposition 7.1 For the retracted local Hamiltonian $\tilde{K}^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ with vanishing trace, the $\det\big(C(\tilde{K}^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0))\big)$ is independent of the choice of the basis $|e_k(\boldsymbol{x}_0)\rangle$ and of the frame $\boldsymbol{\xi}_k$ at $\boldsymbol{x}_0$, and further it is constant on the G-orbit of $\boldsymbol{x}_0$.

8 Local delta-Chern at a degeneracy point

In this section, we show that the local delta-Chern at a degeneracy point is related to the invariant $\det\bigl(C(\tilde{K}^{(2)}_{\mathrm{loc}}(t,q;\boldsymbol{x}_0))\bigr)$.

8.1 Eigen-line bundles for $\tilde{K}^{(2)}_{\mathrm{loc}}(t,q;\boldsymbol{x}_0)$

Since $K(t,q)=\tilde{K}^{(2)}_{\mathrm{loc}}(t,q;\boldsymbol{x}_0)$ is traceless, the degeneracy point for eigenvectors of $K(t,q)$ is determined by

$$\det K(t,q)=0 \quad\Leftrightarrow\quad \boldsymbol{p}(t,q)=C(K(t,q))\begin{pmatrix} t\\ q\end{pmatrix}=0. \tag{94}$$

If $\det C(K(t,q))\neq 0$, the degeneracy point on the tangent plane Π_0 to S^2 at $\boldsymbol{x}_0$ is given by $q=0$ for $t=0$ only. This conclusion is consistent with our assumption that $\boldsymbol{x}_0$ is an isolated degeneracy point corresponding to the parameter value $c(0)$ in the vicinity of $t=0$. The case of $\det C(K(t,q))=0$ will be discussed in Sec. 8.3.

For $t\neq 0$, we can form eigen-line bundles associated with the eigenvalues $\lambda^{\pm}$. According to (25) with $a_{11}-a_{22}=2K_{11}$ and $c_{12}=K_{12}$ (the c_{12} referred to in (25) being different from the c_{12} used in this section), the exceptional point at which the eigenvector for the "up" or "down" eigenvector is not defined is determined in every case by

$$p_2(t,q)=a_{12}t+b_{12}q_1+c_{12}q_2=0,\quad p_3(t,q)=a_{13}t+b_{13}q_1+c_{13}q_2=0. \tag{95}$$

On setting

$$B=B(K(t,q))=\begin{pmatrix} b_{12} & c_{12}\\ b_{13} & c_{13}\end{pmatrix}, \tag{96}$$

we find that if $\det B(K(t,q))\neq 0$, the exceptional point is given by

$$q_1(t)=-\frac{t\begin{vmatrix} a_{12} & c_{12}\\ a_{13} & c_{13}\end{vmatrix}}{\det B},\quad q_2(t)=-\frac{t\begin{vmatrix} b_{12} & a_{12}\\ b_{13} & a_{13}\end{vmatrix}}{\det B}. \tag{97}$$

We here note that if the isotropy subgroup at $\boldsymbol{x}_0$ is non-trivial then one has $a_{12}=a_{13}=0$ from the symmetry condition (56), as is observed on the example (59). In such a case, the exceptional point $q(t)$ is fixed at the origin $q(t)=0$.

In order to assign the exceptional point (97) to "up" or "down" eigenvector, we return to Eq. (25) with $a_{11}-a_{22}=2K_{11}$ and $c_{12}=K_{12}$. For the eigenvector associated with λ^+, according as $K_{11}(t,q(t))>0$ or $K_{11}(t,q(t))<0$, the exceptional point $q(t)$ is assigned to the eigenvector $|u^+_{\mathrm{up}}(t,q)\rangle$ or $|u^+_{\mathrm{down}}(t,q)\rangle$. In contrast with this, for the eigenvector associated with λ^-, according as $K_{11}(t,q(t))<0$ or $K_{11}(t,q(t))>0$, the exceptional point $q(t)$ is assigned to the eigenvector $|u^-_{\mathrm{up}}(t,q)\rangle$ or $|u^-_{\mathrm{down}}(t,q)\rangle$.

The evaluation of K_{11} at the exceptional point (97) gives

$$K_{11}(t,q(t))=\frac{\det C(K(t,q))}{\det B(K(t,q))}t. \tag{98}$$

This means that the sign of K_{11} depends on those of $\det C/\det B$ and t. It then turns out that the domains of respective eigenvectors are given in the following tables:

λ^+	$t<0$	$t>0$
$\det C/\det B > 0$	$U^+_{\text{up}} = \Pi_0$ $U^+_{\text{down}} = \Pi_0 - \{q(t)\}$	$U^+_{\text{up}} = \Pi_0 - \{q(t)\}$ $U^+_{\text{down}} = \Pi_0$
$\det C/\det B < 0$	$U^+_{\text{up}} = \Pi_0 - \{q(t)\}$ $U^+_{\text{down}} = \Pi_0$	$U^+_{\text{up}} = \Pi_0$ $U^+_{\text{down}} = \Pi_0 - \{q(t)\}$

(99)

λ^-	$t<0$	$t>0$
$\det C/\det B > 0$	$U^-_{\text{up}} = \Pi_0 - \{q(t)\}$ $U^-_{\text{down}} = \Pi_0$	$U^-_{\text{up}} = \Pi_0$ $U^-_{\text{down}} = \Pi_0 - \{q(t)\}$
$\det C/\det B < 0$	$U^-_{\text{up}} = \Pi_0$ $U^-_{\text{down}} = \Pi_0 - \{q(t)\}$	$U^-_{\text{up}} = \Pi_0 - \{q(t)\}$ $U^-_{\text{down}} = \Pi_0$

(100)

From (99) and (100), we observe, for example, that in the case of $\det C/\det B < 0$ the exceptional point (97) is assigned to the "down" eigenvector $|u^-_{\text{down}}(t,q)\rangle$ associated with λ^- for $t<0$ and to the "down" eigenvector $|u^+_{\text{down}}(t,q)\rangle$ associated with λ^+ for $t>0$. In particular, for $t=0$, the point $p(t)=0$ becomes the degeneracy point. This observation is depicted in Fig. 4, where x, y are used in place of q_1, q_2, and where the respective (blue) small circles for $t<0$ and $t>0$ are those along which the winding numbers assigned to respective exceptional points is evaluated, as will be shown in what follows.

The transition functions for eigen-line bundles associated with the eigenvalues $\lambda^\pm$ are defined through (73) and given by

$$\Phi^\pm_{(2)}(t,q) = \varepsilon^\pm \frac{X+iY}{\sqrt{X^2+Y^2}}, \quad X = p_2(t,q),\ Y = -p_3(t,q), \tag{101}$$

where $\varepsilon^\pm = \operatorname{sgn}(\lambda^\pm + K_{11})$. The winding number along a small circle γ around the exceptional point $q(t)$ on Π_0 with t fixed is evaluated by

$$\frac{1}{2\pi i}\int_\gamma (\Phi^\pm_{(2)})^{-1} d\Phi^\pm_{(2)}, \tag{102}$$

which is independent of $\varepsilon^\pm$ factor in the definition (101). We here have to be careful in determining the orientation of the circle γ. The orientation of γ should be consistent with the orientation of the corresponding small circle on the sphere S^2. When dividing the sphere into regions S^2_{up} and S^2_{down} by using several circles Γ_j (see Fig. 2), we choose the orientation of Γ_j to be in keeping with the orientation of the region S^2_{up}. Then, a small circle centered at an exceptional point assigned to the "up" eigenvector is clockwise oriented with respect to the positive frame $\boldsymbol{\xi}_k$. In contrast with this, the orientation of a small circle centered at an exceptional point assigned to "down" eigenvector is counterclockwise (see Fig. 3).

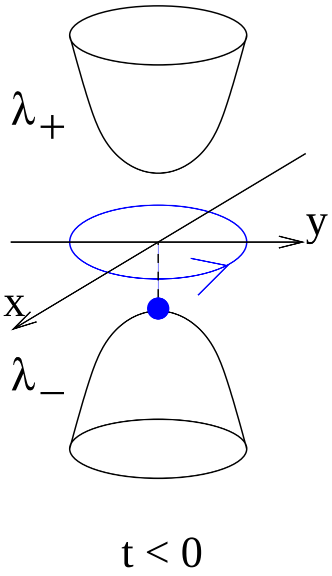

Figure 4: A schematic representation of the evolution of eigenvalues of a local linearized model Hamiltonian in a two-level approximation along with variation of a control parameter t crossing the boundary of the iso-Chern domain. Exceptional points (blue points) assigned to the "down" eigenvector are shown in the λ_+ ($t > 0$) and λ_- ($t < 0$) components.

The winding number assigned to each exceptional point is determined by the sign of

$$\det \begin{pmatrix} \frac{\partial X}{\partial q_1} & \frac{\partial X}{\partial q_2} \\ \frac{\partial Y}{\partial q_1} & \frac{\partial Y}{\partial q_2} \end{pmatrix} \Bigg|_{(q_1(t),q_2(t))} = -\det \begin{pmatrix} b_{12} & c_{12} \\ b_{13} & c_{13} \end{pmatrix} = -\det B(K(t,q)), \tag{103}$$

where $X = p_2(t,q)$, $Y = -p_3(t,q)$. According as the sign of the Jacobian of the left-hand side of the above equation is positive or negative, the winding number of an counterclockwise oriented circle is $+1$ or -1, as was found in Sec. 5.2. If the orientation of the circle is reversed, the sign of the winding number is also reversed.

8.2 Delta-Chern accompanying a change in the parameter

We are now in a position to discuss a change in the winding number assigned to the exceptional point $p(t)$ when the parameter t passes the critical value $t = 0$, and thereby to find a local delta-Chern assigned to the degeneracy point $\boldsymbol{x}_0$. From (99) in the case of the eigen-line bundle associated with λ^+, we observe that according to whether the sign of $\det(C(K))\det(B(K))$ is positive or negative, the exceptional point $p(t)$ is assigned to the "up" eigenvector for $t > 0$ or for $t < 0$. In comparison with this, if the "down" eigenvector is chosen, from the same equation (99) it follows that according to whether the sign of $\det(C(K))\det(B(K))$ is positive or negative, the exceptional point $p(t)$ is assigned to the "down" eigenvector for $t < 0$ or for $t > 0$. We describe the assignment of the exceptional point with the symbol ⋯⋯○—— or ——○⋯⋯, where the small circle stands for the degeneracy point and the solid and broken lines mean that the exceptional point assigned to the "up/down" eigenvector is present and absent, respectively. For example, the symbol ┈┈┈○─── means that there is no exceptional point assigned to the "up/down" eigenvector for $t < 0$, and the exceptional point in question is assigned to the "up/down" eigenvector for $t > 0$.

As is shown in the last subsection (see (103) and the successive sentences), the winding number assigned to an exceptional point is determined from the sign of $\det B$ together with the orientation of the small circle depending on whether it is assigned to "up" or "down" eigenvector. In the case of the "up" eigenvector associated with λ^+, according to whether $\det B(K) > 0$ or $\det B(K) < 0$, the assigned winding number is $+1$ or -1. We denote by $W_{(t>0)}$ and $W_{(t<0)}$ the winding numbers assigned to the exceptional points for $t > 0$ and $t < 0$, respectively. If there is no exceptional point assigned for $t > 0$ or $t < 0$, one has $W_{(t>0)} = 0$ or $W_{(t<0)} = 0$. Thus, if $\det B(K) > 0$, to the symbols ┈┈┈○─── and ───○┈┈┈, we can assign the transitions $W_{(t<0)} = 0 \to W_{(t>0)} = +1$ and $W_{(t<0)} = +1 \to W_{(t>0)} = 0$ in the winding numbers, respectively. If $\det B(K) < 0$, the assigned transitions are $W_{(t<0)} = 0 \to W_{(t>0)} = -1$ and $W_{(t<0)} = -1 \to W_{(t>0)} = 0$, respectively. We define the symbol ΔW by $\Delta W = W_{(t>0)} - W_{(t<0)}$. A similar procedure is performed if the exceptional point assigned to the "down" eigenvector is chosen. Putting all these discussions together, we have the following tables in the case of the eigen-line bundle associated with λ^+.

$\det C(K)$	$\det B(K)$	exc.pt.$_{\text{up}}(t)$	$W_{(t<0)} \to W_{(t>0)}$	ΔW
$+$	$+$	┈┈┈○───	$0 \to +1$	$+1$
$+$	$-$	───○┈┈┈	$-1 \to 0$	$+1$
$-$	$+$	───○┈┈┈	$+1 \to 0$	-1
$-$	$-$	┈┈┈○───	$0 \to -1$	-1

(104)

$\det C(K)$	$\det B(K)$	exc.pt.$_{\text{down}}(t)$	$W_{(t<0)} \to W_{(t>0)}$	ΔW
$+$	$+$	───○┈┈┈	$-1 \to 0$	$+1$
$+$	$-$	┈┈┈○───	$0 \to +1$	$+1$
$-$	$+$	┈┈┈○───	$0 \to -1$	-1
$-$	$-$	───○┈┈┈	$+1 \to 0$	-1

(105)

From the left- and rightmost columns of the above tables, we observe that the variation ΔW in the winding number is given by

$$\Delta W := W_{(t>0)} - W_{(t<0)} = \operatorname{sgn}(\det C(K)), \tag{106}$$

independently of whether the "up" or "down" eigenvector is adopted in evaluating winding numbers. Further, this variation is independent of $\det B(K)$ as well.

In association with the eigenvalue λ^+, we see from (47) and (106) that the variation in the Chern number contribution from a degeneracy point $\boldsymbol{x}_0$, or the local delta-Chern, is given by

$$\Delta^+ c(\boldsymbol{x}_0) = -\operatorname{sgn}(\det C(\tilde{K}^{(2)}_{\text{loc}}(t, q; \boldsymbol{x}_0))), \tag{107}$$

where the superscript $^+$ indicates that the quantity in question is associated with the eigenvalue λ^+. Note that the $\Delta^+ c(\boldsymbol{x}_0)$ is evaluated for the parameter transition from the $t < 0$ side to the $t > 0$ side.

In comparison between (99) and (100), we observe how the exceptional point is assigned to the "up" or "down" eigenvector, and thereby can obtain tables for the eigenvalue λ^- like (104) and (105), from which we see that $\Delta^- c(\boldsymbol{x}_0) = -\Delta^+ c(\boldsymbol{x}_0)$. This is because the sum of the Chern numbers is invariant against the rearrangement of bundle structures.

8.3 The extension of the local delta-Chern formula

In obtaining the local delta-Chern formula (107), we have assumed so far that $\det C(K) \neq 0$ and $\det B(K) \neq 0$. We now consider what happens if $\det C(K) = 0$. Let us assume that $\operatorname{rank} C(K) = 2$. Then, one has $\dim \ker C(K) = 1$. If the one-dimensional vector space $\ker C(K)$ is transverse to the (q_1, q_2) plane in the (t, q_1, q_2)-space, we have a degeneracy point parametrized by t on the (q_1, q_2) plane, by projecting $\ker C(K)$ onto the (q_1, q_2) plane for $t \neq 0$. In this case, eigen-line bundles cannot be defined on the (q_1, q_2) plane. In contrast with this, if $\ker C(K)$ is sitting on the (q_1, q_2) plane, we have a degeneracy line (*i.e.*, a continuum of degeneracy points) on the (q_1, q_2) plane for $t = 0$ only, and no degeneracy point appears on the (q_1, q_2) plane for $t \neq 0$. Then, we may talk about eigen-line bundles on the (q_1, q_2) plane for $t \neq 0$. If $\operatorname{rank} C(K) = 2$ and if for $t \neq 0$ there are no exceptional points emerging from the degeneracy point, or equivalently, if the assignment of exceptional points is described by the symbol $\cdots\cdots\circ\cdots\cdots$, we may have $\Delta^+ c(\boldsymbol{x}_0) = 0$. Here, we note that the condition for no exceptional point for $t \neq 0$ is

$$\operatorname{rank} C_2(K) > \operatorname{rank} B(K), \quad C_2(K) = \begin{pmatrix} a_{12} & b_{12} & c_{12} \\ a_{13} & b_{13} & c_{13} \end{pmatrix}, \ B(K) = \begin{pmatrix} b_{12} & c_{12} \\ b_{13} & c_{13} \end{pmatrix}, \tag{108}$$

as is seen from (95). Hence, the delta-Chern formula is extended so that $\Delta c^+(\boldsymbol{x}_0) = 0 = -\operatorname{sgn}(\det(C(K)))$ may hold, if the definition of sgn is extended so as to be $\operatorname{sgn}(x) = 0$ for $x = 0$.

If $\operatorname{rank} C(K) = 1$, the vector space $\ker C(K)$ is two-dimensional. If it intersects with the plane $t = 0$ along a line, there exists a degeneracy line parametrized by t on the (q_1, q_2)-plane for $t \neq 0$, when $\ker K(C)$ with t fixed is projected there. In this case, eigen-line bundles cannot be defined either. If the $\ker K(C)$ coincides with the (q_1, q_2)-plane, we have a degeneracy plane for $t = 0$ only, and no degeneracy points exist on the (q_1, q_2)-plane for $t \neq 0$. If further Eq. (108) is satisfied, there is no exceptional point for $t \neq 0$. Since $\operatorname{rank} C(K) = 1$, the only possibility for this is that the left-hand and right-hand sides of (108) are 1 and 0, respectively. Then, one has $B(K) = 0$, but the delta-Chern formula may work to result in the vanishing delta-Chern. If both sides of (108) are equal to 1, there exist exceptional points for $t \neq 0$. However, those points form a line, and hence the linearization method fails as well. If both sides of (108) are equal to 0, the linearization method fails of course.

If the linearization method fails at an degeneracy point, we can define the local Hamiltonian by adopting quadratic terms or the first non-vanishing higher-order terms from the expansion of the full Hamiltonian in terms of local coordinates on the tangent plane to S^2 at the degeneracy point. Then, the variation in winding number $\Delta W = W_{(t>0)} - W_{(t<0)}$ can be calculated for the exceptional point of the "up" or "down" eigenvector of the local (quadratic) Hamiltonian, though the second equality of (106) fails. The local delta-Chern assigned to the degeneracy point is now obtained as $-\Delta W$ for the eigen-line bundle associated with the positive (or negative) eigenvalue, as will be mentioned in Sec. 14.1. We will give such an example in Sec. 14.2.

9 Global delta-Chern

We proceed to the global delta-Chern for the full Hamiltonian. Since $\det C(\tilde{K}^{(2)}_{\rm loc}(t,q;\boldsymbol{x}_0))$ is invariant on the G-orbit $\mathcal{O}_{\boldsymbol{x}_0}$ of $\boldsymbol{x}_0$, we have

$$\Delta^+ c(\boldsymbol{x}_0) = \Delta^+ c(g\boldsymbol{x}_0), \quad g\boldsymbol{x}_0 \in \mathcal{O}_{\boldsymbol{x}_0}. \tag{109}$$

In summary, in association with λ^+, the total amount of the delta-Chern throughout the G-orbit is given by

$$\Delta^+ c(\mathcal{O}_{\boldsymbol{x}_0}) = \sum_{g\boldsymbol{x}_0 \in \mathcal{O}_{\boldsymbol{x}_0}} \Delta^+ c(g\boldsymbol{x}_0) = (\#\mathcal{O}_{\boldsymbol{x}_0})\Delta^+ c(\boldsymbol{x}_0). \tag{110}$$

So far we have studied the local delta-Chern in relation to the retract $\tilde{K}^{(2)}_{\rm loc}(t,q;\boldsymbol{x}_0)$ from $K_{\rm loc}(t,q;\boldsymbol{x}_0)$ in normal form with respect to the basis $|e_j(\boldsymbol{x}_0)\rangle$. Our last task is to relate the totality (110) of the local delta-Chern to the global Chern number modification or global delta-Chern for the full Hamiltonian $H(c(t),\boldsymbol{x})$ against the variation in t. Let $K(c(t),\boldsymbol{x})$ denote the Hamiltonian expressed with respect to the basis $|e_j(\boldsymbol{x}_0)\rangle$. From the viewpoint of eigen-line bundles, we may use $K(c(t),\boldsymbol{x})$ as well as $H(c(t),\boldsymbol{x})$. Let Λ_0 be a tangent line to Γ at $c(0)$ in the control parameter space. Then, there exists such a neighborhood of the origin of $\Lambda_0\times\Pi_0$ that it is diffeomorphic to a neighborhood of the point $(c(0),\boldsymbol{x}_0)$ in $\Gamma\times S^2$, where (t,q_1,q_2) serves as a coordinate system on this neighborhood (see the last paragraph of Sec. 7.2). If projected on $\Lambda_0\times\Pi_0$ and expanded in terms of (t,q_1,q_2) in the neighborhood mentioned above, the $K(c(t),\boldsymbol{x})$ can be homotopically deformed to $K_{\rm loc}(t,q;\boldsymbol{x}_0)$ by making higher order terms smaller and then vanish.

We now denote by $|w^\pm_{\rm up/down}(t,\boldsymbol{x})\rangle$ the normalized "up" and "down" eigenvectors associated with the eigenvalues $\nu^\pm(t,\boldsymbol{x})$ of $K(c(t),\boldsymbol{x})$, where $\nu^\pm(t,\boldsymbol{x})$ are supposed to be approximate to the degenerate eigenvalues $\lambda_2(c(0),\boldsymbol{x}_0)=\lambda_3(c(0),\boldsymbol{x}_0)$ in the neighborhood of $(c(0),\boldsymbol{x}_0)\in\Gamma\times S^2$. Then, the setting for constructing the eigen-line bundles for the full Hamiltonian is given by

$$K(c(t),\boldsymbol{x})|w^\pm_{\rm up/down}(t,\boldsymbol{x})\rangle = \nu^\pm(t,\boldsymbol{x})|w^\pm_{\rm up/down}(t,\boldsymbol{x})\rangle, \tag{111}$$

$$|w^\pm_{\rm up}(t,\boldsymbol{x})\rangle = \Phi^\pm_{\rm full}(t,\boldsymbol{x})|w^\pm_{\rm down}(t,\boldsymbol{x})\rangle, \quad \boldsymbol{x}\in W^\pm_{\rm up}\cap W^\pm_{\rm down}, \tag{112}$$

where $\Phi^\pm_{\rm full}(t,\boldsymbol{x})$ are the transition functions and $W^\pm_{\rm up}$ and $W^\pm_{\rm down}$ are the domains of the "up" and "down" eigenvectors, respectively. In a similar manner, the setting for the local Hamiltonian is

$$K_{\rm loc}(t,q;\boldsymbol{x}_0)|v^\pm_{\rm up/down}(t,q)\rangle = \mu^\pm(t,q)|v^\pm_{\rm up/down}(t,q)\rangle, \tag{113}$$

$$|v^\pm_{\rm up}(t,q)\rangle = \Phi^\pm_{(3)}(t,q)|v^\pm_{\rm down}(t,q)\rangle, \quad q\in V^\pm_{\rm up}\cap V^\pm_{\rm down}. \tag{114}$$

Since the $K(c(t),\boldsymbol{x})$ can be homotopically deformed and projected to $K_{\rm loc}(t,q;\boldsymbol{x}_0)$, the eigenvectors $|w^{\pm}_{\rm up/down}(t,\boldsymbol{x})\rangle$ are also homotopically deformed and projected to $|v^{\pm}_{\rm up/down}(t,q)\rangle$, and thereby the transitions function $\Phi^{\pm}_{\rm full}(t,\boldsymbol{x})$ is homotopically deformed and projected to $\Phi^{\pm}_{(3)}(t,q)$ as well. As we have already observed, $\Phi^{\pm}_{(3)}(t,q)$ is approximated by $\Phi^{\pm}_{(2)}(t,q)$, if t,q_1,q_2 are small enough. Thus, we verify that the winding number is deformation invariant:

$$\frac{1}{2\pi i}\int_{\gamma}(\Phi^{\pm}_{\rm full})^{-1}d\Phi^{\pm}_{\rm full}=\frac{1}{2\pi i}\int_{\gamma'}(\Phi^{\pm}_{(3)})^{-1}d\Phi^{\pm}_{(3)}=\frac{1}{2\pi i}\int_{\gamma''}(\Phi^{\pm}_{(2)})^{-1}d\Phi^{\pm}_{(2)}, \tag{115}$$

where γ is a small circle centered at the exceptional point, say, for $|w^{\pm}_{\rm up}(t,\boldsymbol{x})\rangle$ in the neighborhood of $\boldsymbol{x}_0$ in S^2 for either $t>0$ or $t<0$, and where γ' and γ'' are small circles centered at the corresponding exceptional points in the neighborhood of the origin in Π_0.

From (115), we see that the Chern numbers $c^{\pm}$ of the eigen-line bundles associated with the eigenvalues $\nu^{\pm}$ of the full Hamiltonian are given by

$$c^{\pm}=\sum_j\frac{i}{2\pi}\int_{\gamma_j}(\Phi^{\pm}_{\rm full})^{-1}d\Phi^{\pm}_{\rm full}=-\sum_j W^{\pm}(\gamma''_j),\quad W^{\pm}(\gamma''_j)=\frac{1}{2\pi i}\int_{\gamma''_j}(\Phi^{\pm}_{(2)})^{-1}d\Phi^{\pm}_{(2)}. \tag{116}$$

Since the formula (110) is established on the variation (106) in winding numbers, the variation in Chern number for the full Hamiltonian is given by the formula (110) as well.

Theorem 9.1 Assume that accompanying the change in the parameter t from the $t<0$ side to the $t>0$ side, the direct sum of eigen-line bundles for the full Hamiltonian $H(c(t),\boldsymbol{x})$ is reorganized through the degeneracy of eigenvalues, $\lambda_1(c(0),\boldsymbol{x}_0)>\lambda_2(c(0)\boldsymbol{x}_0)=\lambda_3(c(0),\boldsymbol{x}_0)$ at $t=0$, as

$$L_1\oplus L_2\oplus L_3\longrightarrow L_1\oplus L'_2\oplus L'_3. \tag{117}$$

Then, the total (or global) delta-Chern is given by

$$\begin{pmatrix}c(L_1)-c(L_1)\\ c(L'_2)-c(L_2)\\ c(L'_3)-c(L_3)\end{pmatrix}=\overrightarrow{\Delta c}(\mathcal{O}_{\boldsymbol{x}_0})=(\#\mathcal{O}_{\boldsymbol{x}_0})\overrightarrow{\Delta c}(\boldsymbol{x}_0),\quad \overrightarrow{\Delta c}(\boldsymbol{x}_0)=\begin{pmatrix}\Delta^0c(\boldsymbol{x}_0)\\ \Delta^+c(\boldsymbol{x}_0)\\ \Delta^-c(\boldsymbol{x}_0)\end{pmatrix}, \tag{118}$$

where the over-right arrow is attached to indicate that the change in the parameter t is in the positive direction, and where $\Delta^0c(\boldsymbol{x}_0)=0$ and $\Delta^-c(\boldsymbol{x}_0)=-\Delta^+c(\boldsymbol{x}_0)$. If the degeneracy is of the form $\lambda_1(c(0),\boldsymbol{x}_0)=\lambda_2(c(0),\boldsymbol{x}_0)>\lambda_3(c(0),\boldsymbol{x}_0)$, Eqs. (117) and (118) take the form

$$L_1\oplus L_2\oplus L_3\longrightarrow L'_1\oplus L'_2\oplus L_3, \tag{119}$$

$$\begin{pmatrix}c(L'_1)-c(L_1)\\ c(L'_2)-c(L_2)\\ c(L_3)-c(L_3)\end{pmatrix}=\overrightarrow{\Delta c}(\mathcal{O}_{\boldsymbol{x}_0})=(\#\mathcal{O}_{\boldsymbol{x}_0})\overrightarrow{\Delta c}(\boldsymbol{x}_0),\quad \overrightarrow{\Delta c}(\boldsymbol{x}_0)=\begin{pmatrix}\Delta^+c(\boldsymbol{x}_0)\\ \Delta^-c(\boldsymbol{x}_0)\\ \Delta^0c(\boldsymbol{x}_0)\end{pmatrix}, \tag{120}$$

respectively. If the variation in the parameter is reversed, *i.e.*, from the $t>0$ side to the $t<0$ side, the delta-Chern is also reversed through $\overleftarrow{\Delta c}(\boldsymbol{x}_0)=-\overrightarrow{\Delta c}(\boldsymbol{x}_0)$.

As is suggested by the above formula, the formula (118) and (120) can be extended to an n-level system in which only two of eigenvalues are degenerate and the others are distinct, if the G-orbit is a finite set.

Though we have obtained Theorem 9.1 in the case where the linearization method works. However, if the delta-Cherns $\Delta^{\pm}c(\boldsymbol{x}_0)$ can be evaluated, and if the $\Delta^{\pm}c(\boldsymbol{x}_0)$ are constant on the orbit of $\boldsymbol{x}_0$ by G, this theorem holds true. As was mentioned in the last paragraph of Sec. 8.3 and will be explained in Sec. 14.1, we can evaluate the delta-Chern $\Delta^{\pm}c(\boldsymbol{x}_0)$ for a, say, quadratic local Hamiltonian by modifying the study made for the case of linear local Hamiltonian. We will give an example in Sec. 14.2 in which a quadratic local Hamiltonian provides the global delta-Chern.

10 A three-level model with O symmetry

We are to apply the global delta-Chern formula to find out the iso-Chern diagram for the three-level model Hamiltonian given by (19). The first task is to identify the iso-Chern domain by determining degeneracy curves in the control parameter space $\mathbb{R}^2 = \{(a, b)\}$. The next task is to find the Chern numbers of respective eigen-line bundles assigned to one of iso-Chern domains. The third task is to calculate the delta-Chern accompanying the crossing the boundaries of iso-Chern domains by applying Theorem 9.1 with respective local Hamiltonians in normal form. Putting these results together, we can complete the iso-Chern diagram for the Hamiltonian (19) coming from [20].

10.1 Degeneracy points and iso-Chern domains

The first task is to find degeneracy curves in the control parameter space $\mathbb{R}^2 = \{(a, b)\}$. As is already known, the degeneracy points on the sphere S^2 form G-orbits, and the sphere S^2 is stratified into strata according to the action of the symmetry group G. As for the octahedral group, the stratification of the sphere is well known. Each stratum consists of points at which the isotropy subgroup of the O group is isomorphic to a cyclic group C_i with $i = 1, 2, 3, 4$. With respect to the Cartesian coordinate system we adopt on $\mathbb{R}^3$, the strata with isotropy subgroups $C_j, j = 2, 3, 4$, form O-orbits, which are given by (34), (31), (33), respectively. Since these points are entitled to be degeneracy points on S^2, we are in turn allowed to find corresponding degeneracy curves in the control parameter space by using these points. Because of symmetry, we may choose a point from each O-orbit to find such degeneracy curves. The stratum with the trivial isotropy subgroup C_1 will be treated separately.

We first pick up a point $\boldsymbol{x}_0 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ at which the isotropy subgroup is C_2. Then, the Hamiltonian is evaluated at $\boldsymbol{x}_0$ as

$$H(\boldsymbol{x}_0) = \begin{pmatrix} -\frac{a}{2} & \frac{b}{2} & -\frac{i}{\sqrt{2}} \\ \frac{b}{2} & -\frac{a}{2} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & a \end{pmatrix}, \quad \boldsymbol{x}_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}. \tag{121}$$

The characteristic equation for the $H(\boldsymbol{x}_0)$ is written out as

$$\det(\lambda I - H(\boldsymbol{x}_0) = \big(\lambda + \frac{1}{2}(a-b)\big)\Big(\lambda^2 - \frac{1}{2}(a-b)\lambda - \frac{1}{2}(a^2 + ab + 2)\Big) = 0, \tag{122}$$

whose solutions are given by

$$-\frac{1}{2}(a-b), \quad \frac{1}{4}(a-b) \pm \frac{1}{2}\sqrt{\frac{1}{4}(a-b)^2 + 2a(a+b) + 4}. \tag{123}$$

Since the quantity under the square root is positive, degeneracy in eigenvalues occurs between the first eigenvalue and one of the last two. The condition for the degeneracy is shown to be equivalent to

$$b^2 - 2 = 3ab. \tag{124}$$

Hence, when degenerate, the three eigenvalues given above are expressed, independently of $\mathrm{sgn}(b)$, as

$$\frac{1}{3}\big(b + \frac{1}{b}\big), \quad \frac{1}{3}\big(b + \frac{1}{b}\big), \quad -\frac{2}{3}\big(b + \frac{1}{b}\big). \tag{125}$$

This implies that according to whether $b > 0$ or $b < 0$ the degeneracy occurs between upper two eigenvalues or between lower two eigenvalues. Then, the degeneracy occurs in the form $\lambda_1 = \lambda_2 > \lambda_3$ for $b > 0$ or $\lambda_1 > \lambda_2 = \lambda_3$ for $b < 0$.

We turn to a point $\boldsymbol{x}_0 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ at which the isotropy subgroup is C_3. The Hamiltonian takes the form at $\boldsymbol{x}_0$

$$H(\boldsymbol{x}_0) = \begin{pmatrix} 0 & \frac{i}{\sqrt{3}} + \frac{b}{3} & -\frac{i}{\sqrt{3}} + \frac{b}{3} \\ -\frac{i}{\sqrt{3}} + \frac{b}{3} & 0 & \frac{i}{\sqrt{3}} + \frac{b}{3} \\ \frac{i}{\sqrt{3}} + \frac{b}{3} & -\frac{i}{\sqrt{3}} + \frac{b}{3} & 0 \end{pmatrix}, \quad \boldsymbol{x}_0 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}. \tag{126}$$

The characteristic equation is expressed in the factorized form as

$$\det(\lambda I - H(\boldsymbol{x}_0)) = \big(\lambda - \frac{2}{3}b\big)\big(\lambda + \frac{1}{3}b - 1\big)\big(\lambda + \frac{1}{3}b + 1\big) = 0. \tag{127}$$

The eigenvalues are then given by

$$\frac{2}{3}b, \quad -\frac{1}{3}b + 1, \quad -\frac{1}{3}b - 1. \tag{128}$$

Degeneracy in eigenvalues occurs between $\frac{2}{3}b$ and $-\frac{1}{3}b + 1$ for $b > 0$ and between $\frac{2}{3}b$ and $-\frac{1}{3}b - 1$ for $b < 0$. Hence, the degeneracies occur for

$$b = \pm 1. \tag{129}$$

According to whether $b = 1$ or $b = -1$, the eigenvalues are $\lambda_1 = \lambda_2 = \frac{2}{3}, \lambda_3 = -\frac{4}{3}$ or $\lambda_1 = \frac{4}{3}, \lambda_2 = \lambda_3 = -\frac{2}{3}$.

We proceed to a point $\boldsymbol{x}_0 = (0, 0, 1)$ at which the isotropy subgroup is C_4. The Hamiltonian $H(\boldsymbol{x}_0)$ evaluated at $\boldsymbol{x}_0$ is put in the form

$$H(\boldsymbol{x}_0) = \begin{pmatrix} a & i & 0 \\ -i & a & 0 \\ 0 & 0 & -2a \end{pmatrix}, \quad \boldsymbol{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{130}$$

The characteristic equation for $H(\boldsymbol{x}_0)$ is expressed as

$$\det(\lambda I - H(\boldsymbol{x}_0)) = (2a + \lambda)((a - \lambda)^2 + 1) = 0. \tag{131}$$

The eigenvalues are then given by

$$-2a, \quad a \pm 1. \tag{132}$$

Degeneracy of eigenvalues occurs between $-2a$ and $a-1$ for $a > 0$ and between $-2a$ and $a+1$ for $a < 0$. It then follows that the degeneracies occur for the parameter values

$$a = \pm\frac{1}{3}. \tag{133}$$

According to whether $a = \frac{1}{3}$ or $a = -\frac{1}{3}$, the eigenvalues are $\lambda_1 = \frac{4}{3}, \lambda_2 = \lambda_3 = -\frac{2}{3}$ or $\lambda_1 = \lambda_2 = \frac{2}{3}, \lambda_3 = -\frac{4}{3}$.

Thus we have obtained the degeneracy curves (124), (129), (133) in the control parameter space. The remaining task is to consider the O-orbit with the trivial isotropy subgroup C_1. To this end, we can work with the discriminant for the characteristic equation $\det(\lambda I - H(\boldsymbol{x})) = 0$ in terms of invariant polynomials for the O group. The calculation for the discriminant can be made by using computer algebra. As a result, the discriminant $\mathcal{D}(a, b, \boldsymbol{x})$ of the characteristic equation is expressed, in terms of the invariant polynomials for the O group,

$$P_2 = x^2 + y^2 + z^2, \quad P_4 = x^4 + y^4 + z^4, \quad P_6 = x^2y^2z^2, \tag{134}$$

as

$$\begin{aligned}\mathcal{D}(a, b, \boldsymbol{x}) \;=\; & \tfrac{1}{2}(3a-b)^3(3a+b)^3P_4^3 \\ & -\tfrac{3}{4}(135a^4 - 90a^3b - 3a^2b^2 - 144a^2 + 60ab + 12ab^3 - 4b^2 + 36 - 2b^4)(3a+b)^2P_4^2 \\ & +\tfrac{3}{2}(3a-b+2)(3a+b)(3a-b-2)(18a^3 + 6a^2b - 3ab^2 - 12a - 8b - b^3)P_4 \\ & -27(3a-2b)(-ab+3a^2-2)(3a+b)^3P_4P_6 \\ & +27(3a-2b)(5a^3 - ab^2 - 2a - 2b)(3a+b)^2P_6 \\ & -27(3a-2b)^2(3a+b)^4P_6^2 \\ & -\tfrac{1}{4}(3a-b-2)(3a-b+2)(81a^4 + 54a^3b - 36a^2 - 9a^2b^2 - 60ab - 12ab^3 - 2b^4 + 4 - 20b^2)\end{aligned}, \tag{135}$$

where P_2 has been set as $P_2 = 1$ on account of the constraint to the unit sphere. We can show that this discriminant does not vanish on the C_1 stratum by verifying that the discriminant is positive on the C_1 stratum, whereas it vanishes only on the C_j strata with $j = 2, 3, 4$. In order for $\mathcal{D}(a, b, \boldsymbol{x})$ to have a zero at some internal point of the C_1 stratum, the discriminant viewed as a quadratic polynomial in P_6 should vanish, which condition is written as

$$54(3a-2b)^2(3a+b)^4((9P_4-3)a^2 + (1-P_4)b^2 + 2)^3 = 0. \tag{136}$$

Since the range of P_4 is $\frac{1}{3} \le P_4 \le 1$ on the unit sphere, the last factor of the left-hand side of the above equation never vanish. Hence, we have two possibilities for the vanishing of the left-hand side, which are $3a - 2b = 0$ and $3a + b = 0$. (i) In the case of $b = -3a$, the discriminant takes the form

$$\mathcal{D}(a, -3a, \boldsymbol{x}) = 4(3a-1)^2(3a+1)^2. \tag{137}$$

This discriminant vanishes at $a = \frac{1}{3}, b = -1$ and $a = -\frac{1}{3}, b = 1$. We have already obtained these two points in the control parameter space, which are the simultaneous intersection points of the C_2, C_3, and C_4 degeneracy curves and will be discussed in detail later on. (ii) In the case of $3a - 2b = 0$, the discriminant becomes

$$\begin{aligned}
&\mathcal{D}(a, \frac{3a}{2}, \boldsymbol{x}) \\
=&\frac{19683}{128}a^6 P_4^3 + \Big(-\frac{2187}{4}a^2 - \frac{45927}{128}a^6 + \frac{15309}{16}a^4\Big)P_4^2 \\
&+ \Big(\frac{32805}{128}a^6 - \frac{6561}{8}a^4 + 648a^2\Big)P_4 - \frac{6561}{128}a^6 + \frac{2997}{16}a^4 - \frac{693}{4}a^2 + 4.
\end{aligned} \tag{138}$$

As a polynomial in P_4 with the range $\frac{1}{3} \leq P_4 \leq 1$, the above function vanishes only when $P_4 = 1, \frac{1}{2}, \frac{1}{3}$. However, the values $P_4 = 1, \frac{1}{2}, \frac{1}{3}$ are taken only at the orbits whose isotropy subgroups are C_4, C_2, C_3, respectively. It then turns out that there is no degeneracy points with isotropy subgroup C_1.

The degeneracy curves are now shown in Fig. 5, which form boundaries of iso-Chern domains. To each point of degeneracy curves in Fig. 5, there corresponds a set of degeneracy points on S^2, which forms an orbit of the O group. The symbols C_k attached to each degeneracy curve denote the isotropy subgroups at corresponding degeneracy points in S^2.

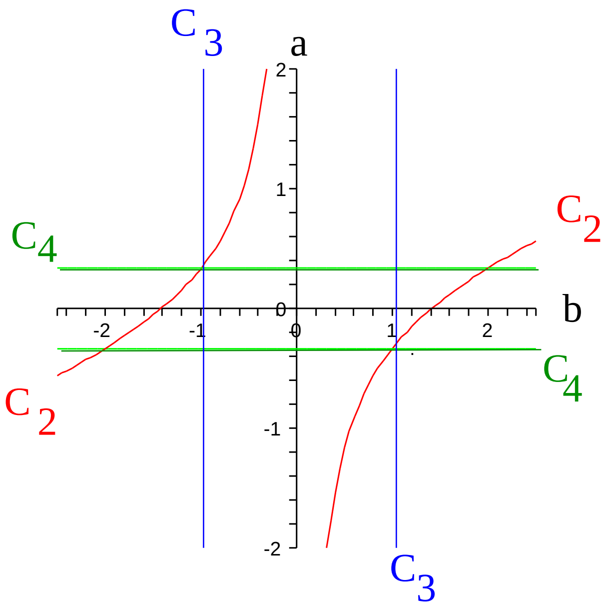

Figure 5: Degeneracy curves in the space of control parameters for the Hamiltonian (19).

On account of Theorem 9.1, we see that the global delta-Chern is given by $\#\mathcal{O}_{\boldsymbol{x}_0} = \#G_0/\#G$ times the local delta-Chern $\Delta^{\pm}c(\boldsymbol{x}_0)$ at a representative degeneracy point $\boldsymbol{x}_0$. If the Chern numbers of the eigen-line bundles assigned to one of iso-Chern domains, the Chern numbers of the eigen-line bundles assigned to every iso-Chern domain are evaluated by using the global delta-Chern formula. Our next tasks are to choose an iso-Chern domain to evaluate the Chern numbers as easily as possible and then to find the local delta-Chern accompanying the crossing of degeneracy curves of every type C_k.

10.2 A seed system

We pick up the iso-Chern domain containing the origin of the control parameter space. The simplest model is given by setting $a = b = 0$,

$$H(\boldsymbol{x}) = \begin{pmatrix} 0 & iz & -iy \\ -iz & 0 & ix \\ iy & -ix & 0 \end{pmatrix}, \quad \boldsymbol{x} \in S^2, \tag{139}$$

which has eigenvalues $\lambda = 0, \pm 1$. For $\lambda = 0$, the normalized eigenvector is given by

$$|w^0(\boldsymbol{x})\rangle = \boldsymbol{x}, \tag{140}$$

which is globally defined on S^2, so that the associated eigen-line bundle is trivial. Hence, the Chern number of this eigen-bundle is 0.

We proceed to the eigen-line bundle associated with $\lambda = 1$. A normalized eigenvector associated with the $\lambda = 1$ is given, in two ways, by

$$|w^+_{\text{up}}(\boldsymbol{x})\rangle = \frac{1}{\sqrt{2(1-z^2)}} \begin{pmatrix} -xz - iy \\ -zy + ix \\ 1 - z^2 \end{pmatrix} \quad \text{on} \quad W^+_{\text{up}} = S^2 - \{z = \pm 1\}, \tag{141a}$$

$$|w^+_{\text{down}}(\boldsymbol{x})\rangle = \frac{1}{\sqrt{2(1-x^2)}} \begin{pmatrix} 1 - x^2 \\ -xy - iz \\ -zx + iy \end{pmatrix} \quad \text{on} \quad W^+_{\text{down}} = S^2 - \{x = \pm 1\}. \tag{141b}$$

The transition function is defined through $|w^+_{\text{up}}(\boldsymbol{x})\rangle = \Phi^+(\boldsymbol{x})|w^+_{\text{down}}(\boldsymbol{x})\rangle$ and given by

$$\Phi^+(\boldsymbol{x}) = \frac{-zx - iy}{\sqrt{(1-z^2)(1-x^2)}} \quad \text{on} \quad W^+_{\text{up}} \cap W^+_{\text{down}}. \tag{142}$$

The local connection forms are defined to be

$$\omega^+_{\text{up}} := \langle w^+_{\text{up}}(\boldsymbol{x})|d|w^+_{\text{up}}(\boldsymbol{x})\rangle, \quad \omega^+_{\text{down}} := \langle w^+_{\text{down}}(\boldsymbol{x})|d|w^+_{\text{down}}(\boldsymbol{x})\rangle \tag{143}$$

on W^+_{up} and on W^+_{down}, respectively, which are related by

$$\omega^+_{\text{up}} = \Phi^+(\boldsymbol{x})^{-1} d\Phi^+(\boldsymbol{x}) + \omega^+_{\text{down}} \quad \text{on} \quad W^+_{\text{up}} \cap W^+_{\text{down}}. \tag{144}$$

The curvature is globally defined by

$$\Omega^+ = \begin{cases} d\omega^+_{\text{up}} & \text{on} \quad W^+_{\text{up}}, \\ d\omega^+_{\text{down}} & \text{on} \quad W^+_{\text{down}}. \end{cases} \tag{145}$$

This system is so simple that we can calculate the Chern number in a straightforward manner. In terms of the spherical polar coordinates (θ, ϕ) on S^2, the local connection form is expressed as

$$\omega_{\rm up}^{+} := \langle w_{\rm up}^{+}(\boldsymbol{x})|d|w_{\rm up}^{+}(\boldsymbol{x})\rangle = i\cos\theta d\phi, \tag{146}$$

and then the curvature as

$$\Omega^{+} = d\omega_{\rm up}^{+} = -i\sin\theta\, d\theta\wedge d\phi, \tag{147}$$

so that we have the Chern number

$$\frac{i}{2\pi}\int_{S^2}\Omega^{+} = \frac{1}{2\pi}\int_{S^2}\sin\theta\, d\theta d\phi = 2. \tag{148}$$

The Chern number of the eigen-line bundle associated with the eigenvalue $\lambda = -1$ should be -2. Thus, we have obtained Chern numbers of respective eigen-line bundles, which are assigned to the iso-Chern domain $\{(a, b) : |a| < \frac{1}{3}, |b| < 1\}$.

10.3 Delta-Chern in crossing C_2 degeneracy curves

We consider the degeneracy curves defined by (124) in the control parameter space. As each point of the degeneracy curves (124) has corresponding degeneracy points on S^2 which form the orbit of the O group with the isotropy subgroup isomorphic to C_2, we refer to the degeneracy curves (124) as C_2 degeneracy curves. As is shown in Fig. 5, the curves have two components, which are distinguished by $b > 0$ and $b < 0$. There are four intersection points with the other degeneracy curves, which are $(a, b) = (\frac{1}{3}, -1), (-\frac{1}{3}, 1)$ and $(a, b) = (\frac{1}{3}, 2), (-\frac{1}{3}, -2)$. As will be seen later, the triple intersection points $(a, b) = (\frac{1}{3}, -1), (-\frac{1}{3}, 1)$ are distinguished from the double intersection points $(a, b) = (\frac{1}{3}, 2), (-\frac{1}{3}, -2)$ and have to be excluded for the reason of the validity of the linearization method.

The Hamiltonian we treat at first is given by (121) together with (124). For $b > 0$, orthonormalized eigenvectors associated with the degenerate eigenvalue $\lambda_1 = \lambda_2 = \frac{1}{3}(b+\frac{1}{b})$ are obtained as

$$|e_1(\boldsymbol{x}_0)_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \quad |e_2(\boldsymbol{x}_0)_+\rangle = \frac{1}{\sqrt{1+b^2}}\begin{pmatrix}-\frac{i}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\\ b\end{pmatrix}, \tag{149}$$

and a normalized eigenvector associated with $\lambda_3 = -\frac{2}{3}(b+\frac{1}{b})$ is given by

$$|e_3(\boldsymbol{x}_0)_+\rangle = \frac{1}{\sqrt{1+b^2}}\begin{pmatrix}\frac{ib}{\sqrt{2}}\\ -\frac{ib}{\sqrt{2}}\\ 1\end{pmatrix}. \tag{150}$$

The eigenvectors $|e_k(\boldsymbol{x}_0)_+\rangle, k = 1, 2, 3$, form an orthonormal basis of $\mathbb{C}^3$, where the subscript $+$ is attached to indicate that $b > 0$.

We turn to the case of $b < 0$. For the degenerate eigenvalue $\lambda_2 = \lambda_3 = \frac{1}{3}(b + \frac{1}{b})$, we have the normalized eigenvectors

$$|e_2(\boldsymbol{x}_0)_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \quad |e_3(\boldsymbol{x}_0)_-\rangle = \frac{1}{\sqrt{1+b^2}}\begin{pmatrix}-\frac{i}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\\ b\end{pmatrix}, \tag{151}$$

and for $\lambda_1 = -\frac{2}{3}(b + \frac{1}{b}) > 0$, we have

$$|e_1(\boldsymbol{x}_0)_-\rangle = \frac{1}{\sqrt{1+b^2}} \begin{pmatrix} \frac{ib}{\sqrt{2}} \\ -\frac{ib}{\sqrt{2}} \\ 1 \end{pmatrix}, \tag{152}$$

where the subscript $-$ is attached in reference to $b < 0$.

We note here that according to the parameter inversion $(a, b) \to (-a, -b)$, the Hamiltonian is subject to the transformation

$$\overline{H(a, b, \boldsymbol{x}_0)} = -H(-a, -b, \boldsymbol{x}_0), \tag{153}$$

where we have denoted the Hamiltonian by $H(a, b, \boldsymbol{x}_0)$ to show its dependence on the parameters a, b. The above equation implies that the eigenvectors $|e_k(\boldsymbol{x}_0)_+\rangle$ and $|e_k(\boldsymbol{x}_0)_-\rangle$ are related by the complex conjugation and the inversion of the parameter b. For example, the complex conjugate of $|e_2(\boldsymbol{x}_0)_+\rangle$ with the inversion $b \to -b$ is equal to $-|e_3(\boldsymbol{x}_0)_-\rangle$.

We now treat the isotropy subgroup $G_0 \cong C_2$ at $\boldsymbol{x}_0$, where $\boldsymbol{x}_0$ is given in (121). The generator of this subgroup is given by

$$h = \begin{pmatrix} & 1 & \\ 1 & & \\ & & -1 \end{pmatrix}. \tag{154}$$

As is easily seen, the representation matrix of h with respect to $|e_k(\boldsymbol{x}_0)_+\rangle$ is expressed as

$$D_+^{(3)}(h) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \tag{155}$$

where the subscript $+$ is attached to indicate that the basis $|e_k(\boldsymbol{x}_0)_+\rangle$ is adopted. In a similar manner, we obtain the representation matrix of h with respect to the basis $|e_k(\boldsymbol{x}_0)_-\rangle$,

$$D_-^{(3)}(h) = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}. \tag{156}$$

Our next task is to take a frame on the tangent plane Π_0 at $\boldsymbol{x}_0$. We here take the frame at $\boldsymbol{x}_0$ as

$$\boldsymbol{\xi}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{\xi}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{157}$$

which is positive in the sense that $\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 = \boldsymbol{x}_0$, where the $\boldsymbol{x}_0$ in the right-hand side of this equation is identified with the unit out-going normal vector at $\boldsymbol{x}_0 \in S^2$. The Cartesian coordinates (q_1, q_2) are introduced on the tangent plane Π_0 through $q_1\boldsymbol{\xi}_1 + q_2\boldsymbol{\xi}_2$. With respect to the frame (157), the action of h on the tangent plane Π_0 is expressed as

$$h^{(2)} = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}. \tag{158}$$

We proceed to calculate the local Hamiltonian defined on the product of the tangent plane Π_0 at $\boldsymbol{x}_0$ and the tangent line Λ_0 at $t = 0$ to a curve (or a line) $c(t)$ transverse to the C_2 degeneracy curve. We take the $c(t)$ as

$$c(t) = (a + t, b), \quad 3ab = b^2 - 2, \quad b \neq \pm 1, \, \pm 2, \tag{159}$$

where t is restricted to such an interval that the line segment $c(t)$ with $|t| < \varepsilon$ may not cross any other degeneracy curves, and where the conditions $b \neq \pm 1, \pm 2$ are imposed so that the line segment $c(t)$ may not pass the intersection points of degeneracy curves. Following the definition (52) of the local Hamiltonian, and expressing the local Hamiltonian with respect to the basis $|e_j(\boldsymbol{x}_0)_+\rangle$ given by (149) and (150) for $b > 0$, we obtain the local Hamiltonian in normal form,

$$\begin{aligned} &K_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)_+ \\ &= \begin{pmatrix} \frac{1}{3}\left(b + \frac{1}{b}\right) - \frac{1}{2}t & \frac{-2(b-1/b)iq_1 + (b^2-1)q_2}{\sqrt{1+b^2}} & \frac{(b^2-3)iq_1 + 2bq_2}{\sqrt{1+b^2}} \\ \frac{2(b-1/b)iq_1 + (b^2-1)q_2}{\sqrt{1+b^2}} & \frac{1}{3}\left(b + \frac{1}{b}\right) + \frac{-\frac{1}{2}+b^2}{1+b^2}t & \frac{3b}{2(1+b^2)}t \\ \frac{-(b^2-3)iq_1 + 2bq_2}{\sqrt{1+b^2}} & \frac{3b}{2(1+b^2)}t & -\frac{2}{3}\left(b + \frac{1}{b}\right) + \frac{1-\frac{1}{2}b^2}{1+b^2}t \end{pmatrix}. \end{aligned} \tag{160}$$

In a similar manner, with respect to the basis $|e_j(\boldsymbol{x}_0)_-\rangle$ given by (151) and (152) for $b < 0$, we obtain the local Hamiltonian in normal form,

$$\begin{aligned} &K_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)_- \\ &= \begin{pmatrix} -\frac{2}{3}\left(b + \frac{1}{b}\right) + \frac{1-\frac{1}{2}b^2}{1+b^2}t & \frac{-(b^2-3)iq_1 + 2bq_2}{\sqrt{1+b^2}} & 0 \\ \frac{(b^2-3)iq_1 + 2bq_2}{\sqrt{1+b^2}} & \frac{1}{3}\left(b + \frac{1}{b}\right) - \frac{1}{2}t & \frac{-2(b-1/b)iq_1 + (b^2-1)q_2}{\sqrt{1+b^2}} \\ 0 & \frac{2(b-1/b)iq_1 + (b^2-1)q_2}{\sqrt{1+b^2}} & \frac{1}{3}\left(b + \frac{1}{b}\right) + \frac{-\frac{1}{2}+b^2}{1+b^2}t \end{pmatrix}. \end{aligned} \tag{161}$$

Retracting the local Hamiltonian to that on the eigenspace associated with the degenerate eigenvalues, *i.e.*, picking up the upper left 2×2 block matrix for $b > 0$ and the lower right 2×2 block matrix for $b < 0$, and making each of those matrices into a traceless matrix, we obtain, in both cases of $b > 0$ and $b < 0$,

$$\tilde{K}^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x})_\pm = \begin{pmatrix} -\alpha_0 t & \alpha_1 q_2 - i\beta_1 q_1 \\ \alpha_1 q_2 + i\beta_1 q_1 & \alpha_0 t \end{pmatrix}, \tag{162}$$

where

$$\alpha_0 = \frac{3b^2}{4(1+b^2)}, \quad \alpha_1 = \frac{b^2-1}{\sqrt{1+b^2}}, \quad \beta_1 = \frac{2(b^2-1)}{b\sqrt{1+b^2}}. \tag{163}$$

Then, the relevant matrices $C(K_\pm)$ and $B(K_\pm)$ defined in (87) and (96), respectively, are expressed as

$$C(K_\pm) = \begin{pmatrix} -\alpha_0 & 0 & 0 \\ 0 & 0 & \alpha_1 \\ 0 & \beta_1 & 0 \end{pmatrix}, \quad B(K_\pm) = \begin{pmatrix} 0 & \alpha_1 \\ \beta_1 & 0 \end{pmatrix}. \tag{164}$$

Since $\det B(K_\pm) \neq 0$ and since

$$\det C(K_\pm) = \alpha_0\alpha_1\beta_1 \begin{cases} > 0 & \text{for} \quad b > 0, \quad b \neq 1, \\ < 0 & \text{for} \quad b < 0, \quad b \neq -1, \end{cases} \tag{165}$$

we find from (107) that

$$\Delta^+ c(\boldsymbol{x}_0) = \begin{cases} -1 & \text{for} \quad b > 0, \quad b \neq 1, \\ +1 & \text{for} \quad b < 0, \quad b \neq -1. \end{cases} \tag{166}$$

It then follows from (118) and (120) with $\#\mathcal{O}_{\boldsymbol{x}_0} = 12$ that the delta-Chern accompanying the variation of the parameter t from the $t < 0$ side to the $t > 0$ side is given by

$$\overrightarrow{\Delta c}(\mathcal{O}_{\boldsymbol{x}_0}) = \begin{cases} \#\mathcal{O}_{\boldsymbol{x}_0} \begin{pmatrix} \Delta^+ c(\boldsymbol{x}_0) \\ \Delta^- c(\boldsymbol{x}_0) \\ \Delta^0 c(\boldsymbol{x}_0) \end{pmatrix} = \begin{pmatrix} -12 \\ 12 \\ 0 \end{pmatrix} & \text{for} \quad b > 0, \quad b \neq 1, \\ \#\mathcal{O}_{\boldsymbol{x}_0} \begin{pmatrix} \Delta^0 c(\boldsymbol{x}_0) \\ \Delta^+ c(\boldsymbol{x}_0) \\ \Delta^- c(\boldsymbol{x}_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 12 \\ -12 \end{pmatrix} & \text{for} \quad b < 0, \quad b \neq -1. \end{cases} \tag{167}$$

We have to make remarks on the case of $b = \pm 1, \pm 2$. As is seen from (163) and (164), if $b = \pm 1$ (or if $(a, b) = (-\frac{1}{3}, 1), (\frac{1}{3}, -1)$), then one has $\mathrm{rank} C(K_\pm) = 1$ and both sides of Eq. (108) vanish; $\mathrm{rank} C_2(K_\pm) = \mathrm{rank} B(K_\pm) = 0$. As was already mentioned in Sec. 8.3, the linearization method fails in these cases. In contrast with this, if $b = \pm 2$ (or if $(a, b) = (\frac{1}{3}, 2), (-\frac{1}{3}, -2)$) in spite of our initial assumption in (159), Eq. (167) is valid, but we have to take into account the formula (209) at the same time for the delta-Chern, since at the points $(a, b) = (\frac{1}{3}, 2), (-\frac{1}{3}, -2)$ isolated degeneracy points exists simultaneously on the C_2 and C_4 degeneracy curves.

10.4 Delta-Chern in crossing C_3 degeneracy curves

We deal with the degeneracy curves defined by (129) in the control parameter space, which we call the C_3 degeneracy curves (or lines). As is done in the preceding section, the triple intersection points $(a, b) = (\frac{1}{3}, -1), (-\frac{1}{3}, 1)$ have to be excluded, but the double intersection points $(a, b) = (\frac{1}{3}, 1), (-\frac{1}{3}, -1)$ are included with due attention. We will make a remark on this problem at the end of the present subsection. The Hamiltonian we start with is given by (126) together with (129). For $b = 1$, orthonormalized eigenvectors associated with the degenerate eigenvalues $\lambda_1 = \lambda_2 = \frac{2}{3}$ are given by

$$|e_1(\boldsymbol{x}_0)_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\pi i/3} \\ 1 \\ 0 \end{pmatrix}, \quad |e_2(\boldsymbol{x}_0)_+\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} e^{-\pi i/3} \\ e^{\pi i/3} \\ 2 \end{pmatrix}, \tag{168}$$

and a normalized eigenvector associated with $\lambda_3 = -\frac{4}{3}$ is given by

$$|e_3(\boldsymbol{x}_0)_+\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{2\pi i/3} \\ -e^{\pi i/3} \\ 1 \end{pmatrix}, \tag{169}$$

where the subscript $+$ is attached to the eigenvectors in reference to $b = 1 > 0$.

Eigenvectors for $b = -1$ are easy to find by using the relation

$$H(\boldsymbol{x}_0)|_{b=-1} = -\overline{H(\boldsymbol{x}_0)}|_{b=1}. \tag{170}$$

It turns out that the normalized eigenvectors of $H(\boldsymbol{x}_0)|_{b=-1}$ associated with the eigenvalue $\lambda_1 = \frac{4}{3}$ is given by

$$|e_1(\boldsymbol{x}_0)_-\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{-2\pi i/3} \\ -e^{-\pi i/3} \\ 1 \end{pmatrix}, \tag{171}$$

and those associated with the degenerate eigenvalues $\lambda_2 = \lambda_3 = -\frac{2}{3}$ are given by

$$|e_2(\boldsymbol{x}_0)_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\pi i/3} \\ 1 \\ 0 \end{pmatrix}, \quad |e_3(\boldsymbol{x}_0)_-\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} e^{\pi i/3} \\ e^{-\pi i/3} \\ 2 \end{pmatrix}, \tag{172}$$

where the subscript $-$ is attached to the eigenvectors in reference to $b = -1 < 0$.

We now treat the action of the isotropy subgroup at the degeneracy point $\boldsymbol{x}_0$ given in (126). The isotropy subgroup G_0 at $\boldsymbol{x}_0$ is isomorphic with the cyclic group C_3 and generated by

$$h = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}. \tag{173}$$

A straightforward calculation shows that with respect to the basis $|e_k(\boldsymbol{x}_0)_+\rangle$ given by (168) and (169) the matrix expression of h takes the form

$$D_+^{(3)}(h) = \begin{pmatrix} \frac{1}{2}e^{\pi i/3} & \frac{\sqrt{3}}{2}e^{-\pi i/3} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2}e^{\pi i/3} & 0 \\ 0 & 0 & e^{-2\pi i/3} \end{pmatrix}. \tag{174}$$

Since the isotropy subgroup $G_0 \cong C_3$ is Abelian, the representation matrix can be of diagonal matrix form with respect to a suitably chosen basis. By introducing the new basis

$$(|e_1'(\boldsymbol{x}_0)_+\rangle, |e_2'(\boldsymbol{x}_0)_+\rangle, |e_3'(\boldsymbol{x}_0)_+\rangle) = (|e_1(\boldsymbol{x}_0)_+\rangle, |e_2(\boldsymbol{x}_0)_+\rangle, |e_3(\boldsymbol{x}_0)_+\rangle)U_+, \tag{175a}$$

$$U_+ = \begin{pmatrix} e^{\pi i/6}/\sqrt{2} & e^{\pi i/6}/\sqrt{2} & 0 \\ ie^{-\pi i/6}/\sqrt{2} & -ie^{-\pi i/6}/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{175b}$$

the representation matrix (174) is diagonalized into

$$\tilde{D}_+^{(3)}(h) = \begin{pmatrix} 1 & & \\ & e^{2\pi i/3} & \\ & & e^{-2\pi i/3} \end{pmatrix}. \tag{176}$$

We turn to the case of $b = -1$. Introducing the new basis by

$$(|e_1'(\boldsymbol{x}_0)_-\rangle, |e_2'(\boldsymbol{x}_0)_-\rangle, |e_3'(\boldsymbol{x}_0)_-\rangle) = (|e_1(\boldsymbol{x}_0)_-\rangle, |e_2(\boldsymbol{x}_0)_-\rangle, |e_3(\boldsymbol{x}_0)_-\rangle)U_-, \tag{177a}$$

$$U_- = \begin{pmatrix} 1 & & \\ & e^{-\pi i/6}/\sqrt{2} & e^{-\pi i/6}/\sqrt{2} \\ & ie^{\pi i/6}/\sqrt{2} & -ie^{\pi i/6}/\sqrt{2} \end{pmatrix}, \tag{177b}$$

we have the representation matrix of the diagonal form

$$\tilde{D}_{-}^{(3)}(h) = \begin{pmatrix} e^{2\pi i/3} & & \\ & e^{-2\pi i/3} & \\ & & 1 \end{pmatrix}. \tag{178}$$

The isotropy subgroup $G_0 \cong C_3$ acts also on the tangent plane Π_0 to S^2 at $\boldsymbol{x}_0$, where $\boldsymbol{x}_0$ is given in (126). We take the frame on Π_0 as

$$\boldsymbol{\xi}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{\xi}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}. \tag{179}$$

This frame is positively oriented, since $\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 = \boldsymbol{x}_0$. The action of h on the plane Π_0 is easily found to be expressed with respect to the basis $\boldsymbol{\xi}_k, k = 1, 2$ as

$$h^{(2)} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}. \tag{180}$$

Our next task is to find the local Hamiltonian in normal form. The C_3 degeneracy curves has two components, which can be distinguished by the sign of b. We take a curve transverse to the C_3 degeneracy curves in the control parameter space as

$$c(t) = \begin{cases} (a, 1+t) & \text{for } b > 0, \\ (a, -1+t) & \text{for } b < 0, \end{cases} \qquad a \neq \pm\frac{1}{3}, \tag{181}$$

where the parameter t is restricted to $|t| < \varepsilon$ so that the curve $c(t)$ may not cross other degeneracy curves in the control parameter space. Following the definition of the local Hamiltonian (52), and expressing the local Hamiltonian with respect to the basis $|e'_j(\boldsymbol{x}_0)_+\rangle$ given by (175) for $b > 0$, we obtain the local Hamiltonian in normal form,

$$K_{\text{loc}}(t, q; \boldsymbol{x}_0)_+ = \begin{pmatrix} \frac{2}{3} + \frac{2}{3}t & i\gamma(q_1 - iq_2) & \beta e^{\pi i/6}(q_1 + iq_2) \\ -i\gamma(q_1 + iq_2) & \frac{2}{3} - \frac{1}{3}t & \gamma e^{\pi i/6}(q_1 - iq_2) \\ \beta e^{-\pi i/6}(q_1 - iq_2) & \gamma e^{-\pi i/6}(q_1 + iq_2) & -\frac{4}{3} - \frac{1}{3}t \end{pmatrix}, \tag{182}$$

where

$$\gamma = \sqrt{2}(a + \frac{1}{3}), \quad \beta = \sqrt{2}(a - \frac{1}{3}). \tag{183}$$

In a similar manner, with respect to the basis $|e'_j(\boldsymbol{x}_0)_-\rangle$ given by (177) for $b < 0$, the local Hamiltonian is expressed as

$$K_{\text{loc}}(t, q; \boldsymbol{x}_0)_- = \begin{pmatrix} \frac{4}{3} - \frac{t}{3} & \beta e^{\pi i/6}(q_1 - iq_2) & \alpha e^{\pi i/6}(q_1 + iq_2) \\ \beta e^{-\pi i/6}(q_1 + iq_2) & -\frac{2}{3} - \frac{t}{3} & i\beta(q_1 - iq_2) \\ \alpha e^{-\pi i/6}(q_1 - iq_2) & -i\beta(q_1 + iq_2) & -\frac{2}{3} + \frac{2}{3}t \end{pmatrix}, \tag{184}$$

where

$$\beta = \sqrt{2}(a - \frac{1}{3}), \quad \alpha = \sqrt{2}(a + \frac{2}{3}). \tag{185}$$

Retracting the local Hamiltonian to that on the eigenspace associated with the degenerate eigenvalues, *i.e.*, picking up the upper left 2×2 block matrix for $b > 0$ and the lower right 2×2 block matrix for $b < 0$, and making each matrix into a traceless matrix, we obtain, in the cases of $b > 0$ and $b < 0$,

$$\tilde{K}^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)_+ = \begin{pmatrix} \frac{t}{2} & i\gamma(q_1 - iq_2) \\ -i\gamma(q_1 + iq_2) & -\frac{t}{2} \end{pmatrix}, \tag{186}$$

$$\tilde{K}^{(2)}_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)_- = \begin{pmatrix} -\frac{t}{2} & i\beta(q_1 - iq_2) \\ -i\beta(q_1 + iq_2) & \frac{t}{2} \end{pmatrix}, \tag{187}$$

respectively. Then, the relevant matrices are written as

$$C(K_+) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & -\gamma & 0 \end{pmatrix}, \quad B(K_+) = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}, \tag{188}$$

$$C(K_-) = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \beta \\ 0 & -\beta & 0 \end{pmatrix}, \quad B(K_-) = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}. \tag{189}$$

Since $\det B(K_\pm) \neq 0$, and since

$$\begin{cases} \det C(K_+) = \frac{1}{2}\gamma^2 > 0 & \text{for} \quad b > 0, \quad a \neq -\frac{1}{3}, \\ \det C(K_-) = -\frac{1}{2}\beta^2 < 0 & \text{for} \quad b < 0, \quad a \neq \frac{1}{3}, \end{cases} \tag{190}$$

we find from (107) that

$$\Delta^+ c(\boldsymbol{x}_0) = \begin{cases} -1 & \text{for} \quad b > 0, \quad a \neq -\frac{1}{3}, \\ +1 & \text{for} \quad b < 0, \quad a \neq \frac{1}{3}. \end{cases} \tag{191}$$

It then follows from (118) and (120) with $\#\mathcal{O}_{\boldsymbol{x}_0} = 8$ that the delta-Chern accompanying the variation of the parameter t from the $t < 0$ side to the $t > 0$ side is given by

$$\overrightarrow{\Delta c}(\mathcal{O}_{\boldsymbol{x}_0}) = \begin{cases} \#\mathcal{O}_{\boldsymbol{x}_0} \begin{pmatrix} \Delta^+ c(\boldsymbol{x}_0) \\ \Delta^- c(\boldsymbol{x}_0) \\ \Delta^0 c(\boldsymbol{x}_0) \end{pmatrix} = \begin{pmatrix} -8 \\ 8 \\ 0 \end{pmatrix} & \text{for} \quad b > 0, \quad a \neq -\frac{1}{3}, \\ \#\mathcal{O}_{\boldsymbol{x}_0} \begin{pmatrix} \Delta^0 c(\boldsymbol{x}_0) \\ \Delta^+ c(\boldsymbol{x}_0) \\ \Delta^- c(\boldsymbol{x}_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ -8 \end{pmatrix} & \text{for} \quad b < 0, \quad a \neq \frac{1}{3}. \end{cases} \tag{192}$$

We note here that for $(a, b) = (-\frac{1}{3}, -1), (\frac{1}{3}, 1)$ the formula (192) is valid in spite of the earlier assumption in (181). However, we have to take into account the formula (209) at the same time, since at the parameter values $(a, b) = (-\frac{1}{3}, -1), (\frac{1}{3}, 1)$ there exist simultaneously isolated degeneracy points on the C_3 and C_4 degeneracy curves. In contrast with this, for $(a, b) = (\frac{1}{3}, -1)$, we have $\mathrm{rank} C(K_-) = 1$ and $\mathrm{rank} C_2(K_-) = \mathrm{rank} B(K_-) = 0$, and for $(a, b) = (-\frac{1}{3}, 1)$, we have $\mathrm{rank} C(K_+) = 1$ and $\mathrm{rank} C_2(K_+) = \mathrm{rank} B(K_+) = 0$. In these cases, the linearization method fails.

10.5 Delta-Chern in crossing C_4 degeneracy curves

We discuss the delta-Chern accompanying the crossing of C_4 degeneracy curves given by (133). We have here to exclude the triple intersection points $(a,b) = (\frac{1}{3}, -1), (-\frac{1}{3}, 1)$ but may include the intersection points $(a,b) = (\frac{1}{3}, 1), (-\frac{1}{3}, -1)$, as in preceding subsections. The curves have two components distinguishable by the sign of a.

We start with the Hamiltonian given by (130) together with (133). For $a = \frac{1}{3}$, orthonormalized eigenvectors of $H(\boldsymbol{x}_0)|_{a=\frac{1}{3}}$ are easily calculated, which are given by

$$|e_1(\boldsymbol{x}_0)_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}, \quad |e_2(\boldsymbol{x}_0)_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}, \quad |e_3(\boldsymbol{x}_0)_+\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{193}$$

where the subscript $+$ is attached in reference to $a = \frac{1}{3} > 0$. The eigenvector $|e_1(\boldsymbol{x}_0)_+\rangle$ is associated with the eigenvalue $\lambda_1 = \frac{4}{3}$, and the eigenvectors $|e_2(\boldsymbol{x}_0)_+\rangle$ and $|e_3(\boldsymbol{x}_0)_+\rangle$ span the eigenspace associated with the doubly degenerate eigenvalue $\lambda_2 = \lambda_3 = -\frac{2}{3}$.

For $a = -\frac{1}{3}$, the eigenvectors are

$$|e_1(\boldsymbol{x}_0)_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}, \quad |e_2(\boldsymbol{x}_0)_-\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |e_3(\boldsymbol{x}_0)_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}, \tag{194}$$

where the subscript $-$ is attached in reference to $a = -\frac{1}{3} < 0$. The eigenvectors $|e_1(\boldsymbol{x}_0)_-\rangle$ and $|e_2(\boldsymbol{x}_0)_-\rangle$ span the eigenspace associated with the doubly degenerate eigenvalue $\lambda_1 = \lambda_2 = \frac{2}{3}$, and the eigenvector $|e_3(\boldsymbol{x}_0)_-\rangle$ is associated with $\lambda_3 = -\frac{4}{3}$.

We discuss the action of the isotropy subgroup $G_0 \cong C_4$ at $\boldsymbol{x}_0$ given in (130). The isotropy subgroup is generated by

$$h = \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix}. \tag{195}$$

The representation matrix of h with respect to the basis $|e_k(\boldsymbol{x}_0)_+\rangle$ is found to be given by

$$D_+^{(3)}(h) = \begin{pmatrix} i & & \\ & -i & \\ & & 1 \end{pmatrix}. \tag{196}$$

Likewise, the representation matrix of h with respect to $|e_k(\boldsymbol{x}_0)_-\rangle$ is given by

$$D_-^{(3)}(h) = \begin{pmatrix} i & & \\ & 1 & \\ & & -i \end{pmatrix}. \tag{197}$$

The isotropy subgroup G_0 at $\boldsymbol{x}_0$ acts also on the tangent plane Π_0 to S^2 at $\boldsymbol{x}_0$. The Π_0 is spanned by the orthonormal basis

$$\boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\xi}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{198}$$

and is endowed with the Cartesian coordinates (q_1, q_2) through $q = q_1\boldsymbol{\xi}_1 + q_2\boldsymbol{\xi}_2$. Since $\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 = \boldsymbol{x}_0$, the tangent plane Π_0 is positively oriented, where $\boldsymbol{x}_0$ in the right-hand side is viewed as a unit normal at $\boldsymbol{x}_0$. The h action on the tangent plan is expressed, with respect to the basis $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$, as

$$h^{(2)} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}. \tag{199}$$

We proceed to discuss the local Hamiltonian on the tangent plane Π_0, taking a curve transverse to the degeneracy curve in the control parameter space by

$$c(t) = \begin{cases} (\frac{1}{3} + t, b) & \text{for} \quad a > 0, \\ (-\frac{1}{3} + t, b) & \text{for} \quad a < 0, \end{cases} \qquad b \neq \pm 1, \pm 2, \tag{200}$$

where t is restricted to a small interval $|t| < \varepsilon$ so that the curve $c(t)$ may not cross the other degeneracy curves. Following the definition of the local Hamiltonian (52), and expressing the local Hamiltonian with respect to the basis $|e_j(\boldsymbol{x}_0)_+\rangle$ given by (193) for $a > 0$, we obtain the local Hamiltonian in normal form,

$$K_{\rm loc}(t, q; \boldsymbol{x}_0)_+ = \begin{pmatrix} \frac{4}{3} + t & 0 & \frac{b-1}{\sqrt{2}}(-iq_1 + q_2) \\ 0 & -\frac{2}{3} + t & \frac{b+1}{\sqrt{2}}(iq_1 + q_2) \\ \frac{b-1}{\sqrt{2}}(iq_1 + q_2) & \frac{b+1}{\sqrt{2}}(-iq_1 + q_2) & -\frac{2}{3} - 2t \end{pmatrix}. \tag{201}$$

In a similar manner, with respect to the basis $|e_j(\boldsymbol{x}_0)_-\rangle$ given by (194) for $a < 0$, the local Hamiltonian in normal form is found to be written as

$$K_{\rm loc}(t, q; \boldsymbol{x}_0)_- = \begin{pmatrix} \frac{2}{3} + t & \frac{b-1}{\sqrt{2}}(-iq_1 + q_2) & 0 \\ \frac{b-1}{\sqrt{2}}(iq_1 + q_2) & \frac{2}{3} - 2t & \frac{b+1}{\sqrt{2}}(-iq_1 + q_2) \\ 0 & \frac{b+1}{\sqrt{2}}(iq_1 + q_2) & -\frac{4}{3} + t \end{pmatrix}. \tag{202}$$

Retracting the local Hamiltonian to that on the eigenspace associated with the degenerate eigenvalues, *i.e.*, picking up the lower right 2×2 block matrix for $a > 0$ and the upper left 2×2 block matrix for $a < 0$, and making respective block matrices into traceless matrices, we obtain, in the cases of $a > 0$ and $a < 0$,

$$\tilde{K}^{(2)}_{\rm loc}(t, q; \boldsymbol{x}_0)_+ = \begin{pmatrix} \frac{3}{2}t & \frac{b+1}{\sqrt{2}}(iq_1 + q_2) \\ \frac{b+1}{\sqrt{2}}(-iq_1 + q_2) & -\frac{3}{2}t \end{pmatrix}, \tag{203}$$

$$\tilde{K}^{(2)}_{\rm loc}(t, q; \boldsymbol{x}_0)_- = \begin{pmatrix} \frac{3}{2}t & \frac{b-1}{\sqrt{2}}(-iq_1 + a_2) \\ \frac{b-1}{\sqrt{2}}(iq_1 + q_2) & -\frac{3}{2}t \end{pmatrix}, \tag{204}$$

respectively. Then, the relevant matrices are expressed as

$$C(K_+) = \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{b+1}{\sqrt{2}} \\ 0 & -\frac{b+1}{\sqrt{2}} & 0 \end{pmatrix}, \quad B(K_+) = \begin{pmatrix} 0 & \frac{b+1}{\sqrt{2}} \\ -\frac{b+1}{\sqrt{2}} & 0 \end{pmatrix}, \tag{205}$$

$$C(K_-) = \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{b-1}{\sqrt{2}} \\ 0 & \frac{b-1}{\sqrt{2}} & 0 \end{pmatrix}, \quad B(K_-) = \begin{pmatrix} 0 & \frac{b-1}{\sqrt{2}} \\ \frac{b-1}{\sqrt{2}} & 0 \end{pmatrix}. \tag{206}$$

Since $\det B(K_\pm) \neq 0$ and since

$$\begin{cases} \det C(K_+) = \frac{3}{2}\left(\frac{b+1}{\sqrt{2}}\right)^2 > 0 & \text{for} \quad a > 0, \quad b \neq -1, \\ \det C(K_-) = -\frac{3}{2}\left(\frac{b-1}{\sqrt{2}}\right)^2 < 0 & \text{for} \quad a < 0, \quad b \neq 1, \end{cases} \tag{207}$$

we find from (107) that

$$\Delta^+ c(\boldsymbol{x}_0) = \begin{cases} -1 & \text{for} \quad a > 0, \quad b \neq -1, \\ +1 & \text{for} \quad a < 0, \quad b \neq 1, \end{cases} \tag{208}$$

It then follows from (118) and (120) with $\#\mathcal{O}_{\boldsymbol{x}_0} = 6$ that the delta-Chern accompanying the variation of the parameter t from the $t < 0$ side to the $t > 0$ side is given by

$$\overrightarrow{\Delta c}(\mathcal{O}_{\boldsymbol{x}_0}) = \begin{cases} \#\mathcal{O}_{\boldsymbol{x}_0} \begin{pmatrix} \Delta^0 c(\boldsymbol{x}_0) \\ \Delta^+ c(\boldsymbol{x}_0) \\ \Delta^- c(\boldsymbol{x}_0) \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ 6 \end{pmatrix} & \text{for} \quad a > 0, \quad b \neq -1, \\ \#\mathcal{O}_{\boldsymbol{x}_0} \begin{pmatrix} \Delta^+ c(\boldsymbol{x}_0) \\ \Delta^- c(\boldsymbol{x}_0) \\ \Delta^0 c(\boldsymbol{x}_0) \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \\ 0 \end{pmatrix} & \text{for} \quad a < 0, \quad b \neq 1. \end{cases} \tag{209}$$

We need remarks similar to those made in the preceding two subsections. In spite of our earlier assumption in (200), the formula (209) is valid for $(a, b) = (\frac{1}{3}, 2), (-\frac{1}{3}, -2)$ and for $(a, b) = (\frac{1}{3}, 1), (-\frac{1}{3}, -1)$. However, for $(a, b) = (\frac{1}{3}, 2), (-\frac{1}{3}, -2)$, we have to take into account the formula (167) at the same time, and for $(a, b) = (\frac{1}{3}, 1), (-\frac{1}{3}, -1)$, the formula (192) at the same time. For the exclusive points $(a, b) = (\frac{1}{3}, -1), (-\frac{1}{3}, 1)$, we have $\mathrm{rank} C(K_+) = 1$, $\mathrm{rank} C_2(K_+) = \mathrm{rank} B(K_+) = 0$, and $\mathrm{rank} C(K_-) = 1$, $\mathrm{rank} C_2(K_-) = \mathrm{rank} B(K_-) = 0$, respectively. In these cases, the linearization method fails.

10.6 Delta-Chern and Chern diagrams

From (167), (192), (209), we have the delta-Chern diagram shown in Fig. 6. Blue vertical lines represent C_3 degeneracy curves (or lines). Green horizontal lines represent C_4 degeneracy curves (or lines). Red curves represent C_2 degeneracy curves. With each degeneracy curve (the boundary of the iso-Chern domain) is associated a three-component column giving delta-Cherns for respective eigen-line bundles with an arrow indicating the direction of the path in the control parameter space. Here, the relation $\overleftarrow{\Delta c}(\mathcal{O}_{\boldsymbol{x}_0}) = -\overrightarrow{\Delta c}(\mathcal{O}_{\boldsymbol{x}_0})$ has been used, depending on which direction the change is made in the parameter t. For example, the upper right column attached to the C_2 degeneracy curve with $b > 0$ is of the form $\overleftarrow{\Delta c}(\mathcal{O}_{\boldsymbol{x}_0}) = \begin{pmatrix} +12 \\ -12 \\ 0 \end{pmatrix} = -\overrightarrow{\Delta c}(\mathcal{O}_{\boldsymbol{x}_0})$, as is seen from (167) with $b > 0$.

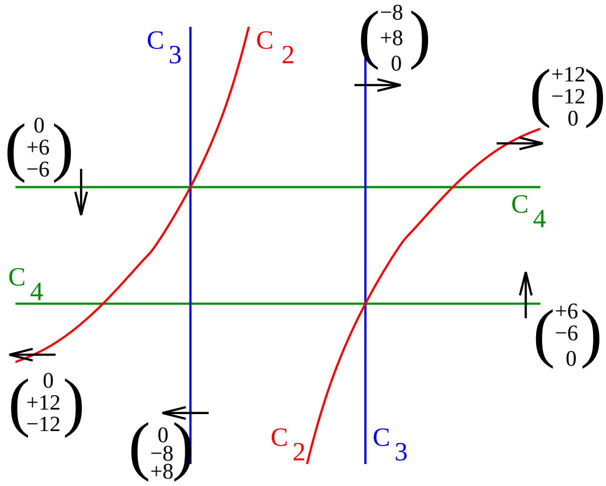

Figure 6: Delta-Chern diagram for the Hamiltonian (19).

Now that we have obtained the delta-Chern diagram, we can find out the iso-Chern diagram as follows: Starting with the column of Chern numbers $\begin{pmatrix} +2 \\ 0 \\ -2 \end{pmatrix}$ assigned to the iso-Chern domain $\{(a, b) : |a| < \frac{1}{3}, |b| < 1\}$, which are obtained in Sec. 10.2, we consecutively apply the delta-Chern diagram to determine the column of Chern numbers on respective iso-Chern domains. For example, starting with the seed column $\begin{pmatrix} +2 \\ 0 \\ -2 \end{pmatrix}$, we obtain

$$\begin{pmatrix} c(L_1) \\ c(L_2') \\ c(L_3') \end{pmatrix} = \begin{pmatrix} +2 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \\ +6 \end{pmatrix} = \begin{pmatrix} +2 \\ -6 \\ +4 \end{pmatrix}, \quad \{(a, b) : |b| < 1, a > \frac{1}{3}, 3ab - b^2 + 2 > 0\}, \quad (210)$$

where $\overleftarrow{\Delta c}(\mathcal{O}_{\boldsymbol{x}_0}) = \begin{pmatrix} 0 \\ +6 \\ -6 \end{pmatrix}$ given by the upper left column attached to a C_4 degeneracy curve in Fig. 6 in used in the reverse form $\overrightarrow{\Delta c}(\mathcal{O}_{\boldsymbol{x}_0})$. Eventually, we obtain the following proposition.

Proposition 10.1 The iso-Chern diagram for the Hamiltonian (19) is shown in Fig. 7. A column of Chern numbers is assigned to each iso-Chern domain bounded by degeneracy curves, where the Chern numbers of the eigen-line bundle associated with the highest, middle, and lowest eigenvalues are placed at the top, middle, and bottom of the column, respectively.

The construction of the whole iso-Chern diagram for our present model has a simple transformation property. The Hamiltonian (19) is subject to the following transformation with respect to the reversing the control parameters (a, b);

$$H(-a, -b, \boldsymbol{x}) = -\overline{H(a, b, \boldsymbol{x})}, \quad (211)$$

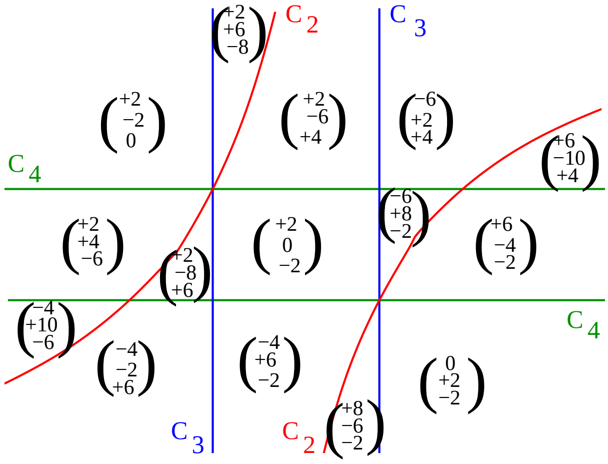

Figure 7: Iso-Chern diagram for the Hamiltonian (19).

where the overline means the complex conjugation. This equation means that changing pair of control parameters (a,b) into $(-a,-b)$ leads first to the overflopping of the band (energy level) order *i.e.*, a band with highest energy becomes a band with lowest energy and vice versa, whereas a band with middle energy remains a middle energy band, and second, to the inversion of the Chern numbers for all bands caused by complex conjugation.

10.7 Adding higher order terms to the initial Hamiltonian

As we have already discussed, there are two special points $(a,b) = (\frac{1}{3},-1),(-\frac{1}{3},1)$ in Fig. 5, at which three degeneracy curves simultaneously cross. As was already observed, for these points, respective local Hamiltonians $\tilde{K}^{(2)}_{\rm loc}(t,q;\boldsymbol{x}_0)$ have degeneracy points everywhere on the tangent plane Π_0 for $t=0$. This conspicuous feature comes from a similar feature of the degeneracy points for the full Hamiltonian. In fact, the set of corresponding degeneracy points is shown to be the whole sphere. As is easily seen, for $(a,b) = (\frac{1}{3},-1),(-\frac{1}{3},1)$, the characteristic equations become independent of $\boldsymbol{x}\in S^2$ and are given by

$$\lambda^3 - \frac{4}{3}\lambda^2 + \frac{16}{27} = 0, \quad \lambda^3 - \frac{4}{3}\lambda^2 - \frac{16}{27} = 0, \tag{212}$$

respectively. Hence, the respective eigenvalues are $\{-\frac{4}{3},\frac{2}{3},\frac{2}{3}\}$ and $\{-\frac{2}{3},-\frac{2}{3},\frac{4}{3}\}$, independently of $\boldsymbol{x}\in S^2$. This means that the corresponding degeneracy points are all the points of the sphere. In this sense, the model Hamiltonian (19) is not generic one.

We can dissolve the triple crossing by adding basis polynomials of degree three to the Hamiltonian (19) in the manner that

$$H(\boldsymbol{x}) + c\begin{pmatrix} 0 & iz(z^2-\frac{3}{5}r^2) & -iy(y^2-\frac{3}{5}r^2) \\ -iz(z^2-\frac{3}{5}r^2) & 0 & ix(x^2-\frac{3}{5}r^2) \\ iy(y^2-\frac{3}{5}r^2) & -ix(x^2-\frac{3}{5}r^2) & 0 \end{pmatrix}, \tag{213}$$

where $r^2=x^2+y^2+z^2=1$, and where c is another control parameter. For the Hamiltonian (213), the control parameter space becomes $\mathbb{R}^3=\{(a,b,c)\}$, and then the degeneracy points form two-dimensional surfaces which are the boundary of iso-Chern domains. By setting $c = \text{const}$, for example $c = \frac{1}{2}$, we restrict the control parameter space to a two-dimensional control parameter space, in which we obtain degeneracy curves forming the boundaries of iso-Chern domains. From the distant point of view, the new degeneracy curves look similar to those given in Fig. 5, but are different from the initial ones in the vicinities of $(a, b) = (-\frac{1}{3}, 1), (\frac{1}{3}, -1)$. In fact, the triple intersections of degeneracy curves disappear and regular intersections of two degeneracy curves come out.

The delta-Chern formula can be applied for the Hamiltonian (213) with $c = \frac{1}{2}$ to yield Fig. 8, in which the vicinity of the point $(a, b) = (-\frac{1}{3}, 1)$ is zoomed in.

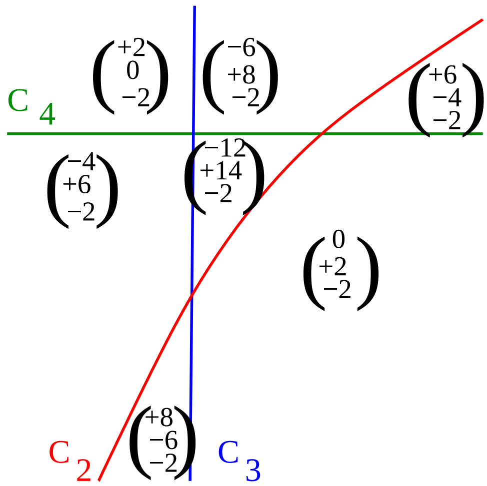

Figure 8: A part of the Chern diagram for the Hamiltonian (213) with $c = \frac{1}{2}$ is zoomed in.

11 The two-level model revisited

The delta-Chern formula (107) may be applied to the two-level system treated in Sec. 5. For this system, we have two types of degeneracy points, $(a, b) = (\pm 1, 0), (0, \pm 1)$, on the reduced control parameter space, the unit circle $a^2 + b^2 = 1$. The former points $(\pm 1, 0)$ are called C_3 degeneracy points and the latter C_1 degeneracy points, since the former have the corresponding degeneracy points at each of which the isotropy subgroup is isomorphic to C_3 and the latter have the corresponding degeneracy points at each of which the isotropy subgroup is C_1 in general. We will encounter the case where we have to use the extended local-Chern formula discussed in Sec. 8.3. Using this model as an example, we also show that the appearance of exceptional points depends on the choice of a basis with respect to which the Hamiltonian is expressed but the Chern number is independent of the appearance of exceptional points.

11.1 Delta-Chern in crossing C_3 degeneracy points

At a C_3 degeneracy point $\boldsymbol{x}_0$, the Hamiltonian (28) is evaluated as

$$H(\boldsymbol{x}_0) = \begin{pmatrix} 0 & -\frac{ib}{3\sqrt{3}} \\ \frac{ib}{3\sqrt{3}} & 0 \end{pmatrix}, \quad \boldsymbol{x}_0 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}. \tag{214}$$

The eigenvalues of $H(\boldsymbol{x}_0)$ are $\mu = \pm\frac{b}{3\sqrt{3}}$. When $b = 0$, these eigenvalues are degenerate, and $H(\boldsymbol{x}_0)$ becomes a zero matrix, so that any linearly independent vectors serve as associated eigenvectors. In order to find a suitable choice of eigenvectors, we try to take normalized eigenvectors associated with

$$\mu_1 = \frac{b}{3\sqrt{3}}, \quad \mu_2 = -\frac{b}{3\sqrt{3}}, \tag{215}$$

which are found to be given by

$$|e_1(\boldsymbol{x}_0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}, \quad |e_2(\boldsymbol{x}_0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}, \tag{216}$$

respectively. Since these eigenvectors are independent of b, we are allowed to take them as basis vectors of the eigenspace associated with the degenerate eigenvalue for $b = 0$.

We now treat the action of the isotropy subgroup at the degeneracy point $\boldsymbol{x}_0$. The isotropy subgroup at $\boldsymbol{x}_0$ is isomorphic with C_3, and generated by h given in (173). In the two-dimensional irreducible representation of the total group O, the h is known to be represented as

$$D(h) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}. \tag{217}$$

As is easily verified, the representation matrix of h is put with respect to the basis $|e_1(\boldsymbol{x}_0)\rangle$ and $|e_2(\boldsymbol{x}_0)\rangle$ in the form

$$\tilde{D}(h) = \begin{pmatrix} e^{2\pi i/3} & \\ & e^{-2\pi i/3} \end{pmatrix}. \tag{218}$$

We now consider the action of the $h \in C_3$ on the tangent plane Π_0 to S^2 at $\boldsymbol{x}_0$. The frame $\boldsymbol{\xi}_k$ is the same as given in (179), and the representation matrix of h with respect to the basis $\boldsymbol{\xi}_k$ is given by (180).

We proceed to the local Hamiltonian on the tangent plane Π_0, taking the curve in the control parameter space,

$$c(t) = (1, t), \tag{219}$$

which is viewed as a tangent line to the unit circle $a^2 + b^2 = 1$ at $(a, b) = (1, 0)$. The local Hamiltonian expressed with respect to the basis $|e_k(\boldsymbol{x}_0)\rangle$ given in (216) is written out as

$$K_{\mathrm{loc}}(t, q; \boldsymbol{x}_0) = \begin{pmatrix} \frac{t}{3\sqrt{3}} & -2\sqrt{2}i(q_1 - iq_2) \\ 2\sqrt{2}i(q_1 + iq_2) & -\frac{t}{3\sqrt{3}} \end{pmatrix}, \tag{220}$$

which is in normal form. The relevant matrices are

$$C(K) = \begin{pmatrix} \frac{1}{3\sqrt{3}} & 0 & 0 \\ 0 & 0 & -2\sqrt{2} \\ 0 & 2\sqrt{2} & 0 \end{pmatrix}, \quad B(K) = \begin{pmatrix} 0 & -2\sqrt{2} \\ 2\sqrt{2} & 0 \end{pmatrix}. \tag{221}$$

Since $\det B(K) \neq 0$, and since $\det C(K) = \frac{8}{3\sqrt{3}} > 0$, one has, from (107),

$$\Delta^+ c(\boldsymbol{x}_0) = -1. \tag{222}$$

Since $\#\mathcal{O}_{\boldsymbol{x}_0} = 8$, we obtain, from (110),

$$\Delta^+ c(\mathcal{O}_{\boldsymbol{x}_0}) = -8. \tag{223}$$

As is seen in Fig. 1, this explains the delta-Chern observed when passing the degeneracy point $(a, b) = (1, 0)$ from the $t < 0$ side to the $t > 0$ side; $c(L_1') - c(L_1) = \Delta^+ c(\mathcal{O}_{\boldsymbol{x}_0})$ with $c(L_1') = -4$ and $c(L_1) = 4$.

11.2 Delta-Chern in crossing C_1 degeneracy points

We take the degeneracy point $(a, b) = (0, 1)$ and the tangent line

$$c(t) = (t, 1) \tag{224}$$

to the unit circle $a^2 + b^2 = 1$ at $(a, b) = (0, 1)$. The corresponding degeneracy points on the sphere S^2 have been given in (32). We take a degeneracy point $\boldsymbol{x}_0$ sitting on the circle $x^2 + y^2 = 1$ but away from the orbits of type $C_r, r = 4, 2$,

$$\boldsymbol{x}_0 = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}, \quad \varphi \neq k\frac{\pi}{4}, \quad k = 0, 1, 2, \ldots, 7. \tag{225}$$

The isotropy subgroup at $\boldsymbol{x}_0$ is trivial, that is, isomorphic to C_1. The Hamiltonian evaluated at $\boldsymbol{x}_0$ is then written as

$$H(\boldsymbol{x}_0) = \begin{pmatrix} -t & \sqrt{3}t\cos 2\varphi \\ \sqrt{3}t\cos 2\varphi & t \end{pmatrix}. \tag{226}$$

We choose the frame $\boldsymbol{\xi}_k$ at $\boldsymbol{x}_0$ as follows:

$$\boldsymbol{\xi}_1 = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \quad \boldsymbol{\xi}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{227}$$

This frame is positively oriented, since $\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 = \boldsymbol{x}_0$, where the $\boldsymbol{x}_0$ in the right-hand side is viewed as the outgoing unit normal at $\boldsymbol{x}_0 \in S^2$.

The local Hamiltonian is now given by

$$H_{\text{loc}}(t, q; \boldsymbol{x}_0) = \begin{pmatrix} -t & \sqrt{3}t\cos 2\varphi - \frac{1}{2}i\sin 2\varphi\, q_2 \\ \sqrt{3}t\cos 2\varphi + \frac{1}{2}i\sin 2\varphi\, q_2 & t \end{pmatrix}. \tag{228}$$

Since the isotropy subgroup is trivial, this local Hamiltonian is already in normal form, so that $\tilde{H}_{\mathrm{loc}} = \tilde{K}_{\mathrm{loc}}$. The relevant matrices are expressed as

$$C(K) = \begin{pmatrix} -1 & 0 & 0 \\ \sqrt{3}\cos 2\varphi & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sin 2\varphi \end{pmatrix}, \quad B(K) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\sin 2\varphi \end{pmatrix}, \quad \varphi \neq k\frac{\pi}{4}, \tag{229}$$

where rank$C(K) = 2$ and rank$B(K) = 1$. Further, since

$$\mathrm{rank}\begin{pmatrix} \sqrt{3}\cos 2\varphi & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sin 2\varphi \end{pmatrix} > \mathrm{rank}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\sin 2\varphi \end{pmatrix}, \tag{230}$$

the inequality (108) is satisfied, so that no exceptional point appears for $t \neq 0$. Hence, we conclude that

$$\Delta^{+} c(\boldsymbol{x}_0) = -\det C(K) = 0, \tag{231}$$

which explains the delta-Chern observed when passing the degeneracy point $(a, b) = (0, 1)$ from the $t < 0$ side to the $t > 0$ side; $c(L_1') - c(L_1) = \Delta^{+} c(\mathcal{O}_{\boldsymbol{x}_0})$ with $c(L_1') = -4$ and $c(L_1) = -4$.

We here make a comment on the fact that the degeneracy point is not isolated. Although the set of degeneracy points is not a finite set but a continuum, it is a disjoint union of finite sets from a viewpoint of orbits,

$$\bigsqcup_{0<\varphi<\frac{\pi}{4}} \bigcup_{g\in O} \left\{ g \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \right\}. \tag{232}$$

For this reason, we are allowed to consider a degeneracy point $\boldsymbol{x}_0$ irrespective of the other degeneracy points in order to discuss the delta-Chern.

To make the present discussion on the delta-Chern complete, we have to treat the embedded degeneracy points, which correspond to $\varphi = k\frac{\pi}{4}$. We take a degeneracy point

$$\boldsymbol{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{233}$$

which corresponds to $\varphi = 0$. The isotropy subgroup at the present $\boldsymbol{x}_0$ is generated by

$$h = \begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}, \tag{234}$$

and its representation matrix with respect to the E representation takes the from

$$D(h) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}. \tag{235}$$

From (228) with $\varphi = 0$, the local Hamiltonian is given by

$$H_{\mathrm{loc}}(t, q; \boldsymbol{x}_0) = \begin{pmatrix} -t & \sqrt{3}t \\ \sqrt{3}t & t \end{pmatrix}. \tag{236}$$

To put this local Hamiltonian in normal form, we take a new basis forming the unitary matrix

$$U_2 = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}. \tag{237}$$

Then, we obtain the local Hamiltonian in normal form

$$K_{\rm loc}(t, q : \boldsymbol{x}_0) = \begin{pmatrix} -t & -\sqrt{3}t \\ -\sqrt{3}t & t \end{pmatrix}. \tag{238}$$

The relevant matrices are then expressed as

$$C(K) = \begin{pmatrix} -1 & 0 & 0 \\ -\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B(K) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{239}$$

Since the inequality (108) is satisfied, there is no exceptional point for $t \neq 0$. Hence, the delta-Chern vanishes, as is expected.

We turn to anther degeneracy point $\boldsymbol{x}_0 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, which was treated in Sec. 10.3. A generator of the isotropy subgroup at $\boldsymbol{x}_0$ and its representation matrix are given by

$$h = \begin{pmatrix} & 1 & \\ 1 & & \\ & & -1 \end{pmatrix}, \quad D(h) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \tag{240}$$

respectively. Since $D(h)$ is already of diagonal form, the local Hamiltonian given in (228) with $\varphi = \frac{\pi}{4}$ provides the local Hamiltonian in normal form,

$$K_{\rm loc}(t, q; \boldsymbol{x}_0) = \begin{pmatrix} -t & -\frac{i}{2}q_2 \\ \frac{i}{2}q_2 & t \end{pmatrix}. \tag{241}$$

The relevant matrices are then expressed as

$$C(K) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad B(K) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{242}$$

Because of the existence of a continuum of exceptional points $q_2 = 0$, the linearization method says nothing about the delta-Chern, where $\mathrm{rank} C_2(K) = \mathrm{rank} B(K) = 1$ and then the inequality (108) is not satisfied.

11.3 Basis dependence of the appearance of exceptional points

In Sec. 11.1, we have taken the basis (216) in putting the local Hamiltonian in normal form. We are now interested in treating the full Hamiltonian expressed with respect to this basis, which is written as

$$K(\boldsymbol{x}) = \begin{pmatrix} b\phi_3 & -a(\phi_1 - i\phi_2) \\ -a(\phi_1 + i\phi_2) & -b\phi_3 \end{pmatrix}, \tag{243}$$

where we have used the symbol K in order to indicate that the basis (216) has been adopted. The degeneracy points in the control parameter space and those on the sphere S^2 are, of course, the same as those for the initial Hamiltonian $H(\boldsymbol{x})$ (see Prop. 5.1 and (31)). As for exceptional points, if $a \neq 0$, they are determined by and assigned to the "up" and "down" eigenvectors according to

$$\phi_1 = \phi_2 = 0, \quad b\phi_3 > 0, \tag{244a}$$

$$\phi_1 = \phi_2 = 0, \quad b\phi_3 < 0, \tag{244b}$$

respectively. It then turns out that if $b > 0$ the exceptional points assigned to the "up" eigenvector associated with the positive eigenvalue λ^+ are only

$$\boldsymbol{e}_1^+ = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \ \boldsymbol{e}_2^+ = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \ \boldsymbol{e}_3^+ = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}, \ \boldsymbol{e}_4^+ = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}, \tag{245}$$

and the exceptional points assigned to the "down" eigenvector associated with the λ^+. are only

$$\boldsymbol{e}_1^- = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}, \ \boldsymbol{e}_2^- = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \ \boldsymbol{e}_3^- = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}, \ \boldsymbol{e}_4^- = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}. \tag{246}$$

If $b < 0$, the $\{\boldsymbol{e}_j^-\}_{j=1}^4$ are assigned to the "up" eigenvector and the $\{\boldsymbol{e}_j^+\}_{j=1}^4$ to the "down" eigenvector. The manifestation of the exceptional points $\{\boldsymbol{e}_j^+\}_{j=1}^4$ or $\{\boldsymbol{e}_j^-\}_{j=1}^4$ for $K(\boldsymbol{x})$ is in marked contrast with that of $\{\boldsymbol{n}_\pm\}$ or $\{\boldsymbol{a}_\pm, \boldsymbol{b}_\pm\}$ for $H(\boldsymbol{x})$ (see (35)). Thus, the appearance of exceptional points depends on the choice of the bases with respect to which the Hamiltonian is expressed. We note in addition that each of $\{\boldsymbol{e}_j^+\}_{j=1}^4$ and $\{\boldsymbol{e}_j^-\}_{j=1}^4$ is the orbit of the tetrahedral group (the group of rotational symmetries of a regular tetrahedron), a subgroup of the octahedral group.

The Chern number can be found by evaluating the winding number assigned to each of the exceptional points $\{\boldsymbol{e}_j^+\}_{j=1}^4$ or to each of $\{\boldsymbol{e}_j^-\}_{j=1}^4$, like (47). Among the exceptional points $\{\boldsymbol{e}_j^+\}_{j=1}^4$ to be assigned to the "up" eigenvector for $b > 0$, we pick up $\boldsymbol{e}_1^+$, which is the same as $\boldsymbol{x}_0$ given in (214). We adopt the linearization method explained in Sec. 5.2. We then look at the local Hamiltonian (220) with t replaced by b. From this Hamiltonian, we can obtain the winding number assigned to $\boldsymbol{e}_1^+$. Since the constant factor is irrelevant to the winding number, we see from the $(1, 2)$ component of (220) that the quantity $q_1 - iq_2$ determines the winding number together with the orientation convention of a small circle centered at $\boldsymbol{e}_1^+$. Since the orientation of the circle is clockwise for the exceptional point of the "up" eigenvector, the winding number in question is $+1$, and hence the Chern number contribution from $\boldsymbol{e}_1^+$ is -1. The same calculation can apply to all of $\{\boldsymbol{e}_j^+\}_{j=1}^4$, and hence we find that the Chern number for $b > 0$ and $a \neq 0$ is -4.

We now pick up the point $\boldsymbol{e}_2^-$, which we denote by $\boldsymbol{x}_0'$. The $\boldsymbol{x}_0$ and $\boldsymbol{x}_0'$ are related by

$$\boldsymbol{x}_0' = g\boldsymbol{x}_0, \quad g = \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix}. \tag{247}$$

In the E representation of the O group, the representation matrix of g is written as

$$D(g) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \tag{248}$$

We take the basis $|e_k(\boldsymbol{x}_0')\rangle$ and the frame $\boldsymbol{\xi}_k'$ on the tangent plane $\Pi_{\boldsymbol{x}_0'}$ as

$$|e_k(\boldsymbol{x}_0')\rangle = D(g)|e_k(\boldsymbol{x}_0)\rangle, \quad \boldsymbol{\xi}_k' = g\boldsymbol{\xi}_k, \tag{249}$$

respectively. Then, the local Hamiltonian in normal from at $\boldsymbol{x}_0'$ is the same as (220); $K_{\mathrm{loc}}(t,q;\boldsymbol{x}_0') = K_{\mathrm{loc}}(t,q;\boldsymbol{x}_0)$ with t replaced by b. For $b<0$, the exceptional point $\boldsymbol{x}_0'$ is assigned to the "down" eigenvector. Hence, the small circle centered at $\boldsymbol{x}_0'$ is oriented anticlockwise. With this orientation taken into account, the winding number assigned to $\boldsymbol{x}_0'$ is determined by $q_1 - iq_2$ from the $(1,2)$-component of $K_{\mathrm{loc}}(t,q;\boldsymbol{x}_0')$, which is -1. Hence, the Chern number contribution from $\boldsymbol{e}_2^-$ is $+1$. The same calculation can apply to all of $\{\boldsymbol{e}_j^-\}_{j=1}^4$, and thereby the Chern number for $b<0$ and $a \neq 0$ is $+4$. Thus we have obtained again Fig. 1 by following the same method as in Sec. 5.2 with exceptional points other than those adopted in Sec. 5.2.

Summing up the above discussion, we obtain the following tables on calculation of the Chern number for the eigen-line bundle associate with the positive eigenvalue λ^+;

"up"	$b<0$	$b>0$
$\{\boldsymbol{e}_j^+\}_{j=1}^4$	⋯⋯⋯⋯	○————
$\{\boldsymbol{e}_j^-\}_{j=1}^4$	————○	⋯⋯⋯⋯
Chern no.	$+4$	-4

"down"	$b<0$	$b>0$
$\{\boldsymbol{e}_j^+\}_{j=1}^4$	————○	⋯⋯⋯⋯
$\{\boldsymbol{e}_j^-\}_{j=1}^4$	⋯⋯⋯⋯	○————
Chern no.	$+4$	-4

(250)

Here, for example, the symbol ⋯⋯⋯○———— means that for $b<0$ the points in the leftmost column are not assigned as exceptional points but assigned for $b>0$. Thus, the availability of the linearization at an exceptional point depends on the choice of the basis with respect to which the Hamiltonian is expressed.

11.4 Basis dependence of the local Hamiltonian

We here recall that $\det C(K)$ is an invariant, but not so is $\det B(K)$. Though the delta-Chern formula is independent of $\det B(K)$, we are interested in basis dependence of $\det B(K)$, which leads to a better understanding of the (t,q_1,q_2) space.

We now return to the local Hamiltonian (220), which is in normal form. In contrast with this, the local Hamiltonian

$$H_{\mathrm{loc}}(t,q;\boldsymbol{x}_0) = \begin{pmatrix} 2\sqrt{2}q_2 & -2\sqrt{2}q_1 - i\frac{t}{3\sqrt{3}} \\ -2\sqrt{2}q_1 + i\frac{t}{3\sqrt{3}} & -2\sqrt{2}q_2 \end{pmatrix}, \tag{251}$$

which is expressed with respect to the standard basis, is not in normal form. The relevant matrices to this local Hamiltonian are written as

$$C(H) = \begin{pmatrix} 0 & 0 & 2\sqrt{2} \\ 0 & -2\sqrt{2} & 0 \\ \frac{1}{3\sqrt{3}} & 0 & 0 \end{pmatrix}, \quad B(H) = \begin{pmatrix} -2\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}. \tag{252}$$

For $B(H)$, we have $\det B(H) = 0$, and thereby the quantity corresponding to (98) makes no sense. A question now arises as to whether the local delta-Chern formula (107), which has been obtained on the assumption that $\det B(K) \neq 0$, is available even if $\det B(H) = 0$ or not. We infer that the delta-Chern formula should hold, since $\det C(H) = \det C(K)$ and since the formula is independent of $\det B(K)$.

For the local Hamiltonian (251), the degeneracy point is given by $q_1 = q_2 = t = 0$. If we formally apply the procedure (25), we conclude that the exceptional points determined by $t = 0, q_1 = 0$ for the positive eigenvalue of (251) are assigned to "up" or "down" eigenvectors according as $q_2 > 0$ or $q_2 < 0$. However, this conclusion sounds strange. In fact, although the exceptional points of the eigenvector for the full Hamiltonian are isolated points, which are given by (35), the exceptional points in question form a half line on the (q_1, q_2) plane. It then seems that the local Hamiltonian (251) fails to work for the delta-Chern formula.

We now show that this discrepancy comes from our interpretation of the coordinates (t, q_1, q_2). We think of t as a control parameter and of (q_1, q_2) as locally-defined dynamical variables. This interpretation is reasonable from the physical point of view. However, we have to be reminded of the fact that a change of basis with respect to which the Hamiltonian is expressed gives rise to a rotation on the (t, q_1, q_2) space, which is shown in (81). This implies that the distinguishing of (q_1, q_2) from t is a concept depending on the choice of bases. With the basis (216), we associated the unitary matrix

$$U_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}. \tag{253}$$

Then, two expressions of the local Hamiltonian are related by

$$U_2^{-1} H_{\text{loc}}(t, q; \boldsymbol{x}_0) U_2 = K_{\text{loc}}(t, q; \boldsymbol{x}_0). \tag{254}$$

The induced rotation $h = (h_{jk})$ defined through $U_2^{-1} \sigma'_j U_2 = \sum_k h_{kj} \sigma'_k$ has the representation matrix

$$h = \begin{pmatrix} & & 1 \\ -1 & & \\ & -1 & \end{pmatrix}, \tag{255}$$

which transforms the set of seemingly strange exceptional points $\{(0, 0, \tau);\ \tau > 0\}$ for the "up" eigenvector of $H_{\text{loc}}(t, q; \boldsymbol{x}_0)$ into the set of reasonable exceptional points $\{(\tau, 0, 0); \tau > 0\}$ for the "up" eigenvector of $K_{\text{loc}}(t, q; \boldsymbol{x}_0)$. Hence, we are allowed to say that we choose to use the local Hamiltonian in normal form in search for the delta-Chern in order to make distinct the difference between the physical variables and the control parameter.

12 A case study for triply degenerate eigenvalues

So far we have treated doubly degenerate eigenvalues and obtained the delta-Chern formula accompanying the crossing of a degeneracy curve corresponding to doubly degenerate eigenvalues.

We now wish to observe what happens in the delta-Chern formula if triple degeneracy occurs in eigenvalues, by using a model Hamiltonian. The model Hamiltonian we take up here is given by

$$H(\boldsymbol{x}) = \begin{pmatrix} 0 & -iZ & iY \\ iZ & 0 & -iX \\ -iY & iX & 0 \end{pmatrix}, \quad \boldsymbol{x} \in S^2 \subset \mathbb{R}^3, \tag{256}$$

where X, Y, Z are functions defined to be

$$X = ax + bx(x^2 - \frac{3}{5}r^2), \tag{257a}$$

$$Y = ay + by(y^2 - \frac{3}{5}r^2), \tag{257b}$$

$$Z = az + bz(z^2 - \frac{3}{5}r^2), \tag{257c}$$

respectively, where $r^2 = x^2+y^2+z^2$, and where a, b are real parameters with $(a, b) \neq (0, 0)$. The constraint $r = 1$ is imposed, of course. This Hamiltonian is a special one from (213).

As is easily verified, the eigenvalues of $H(\boldsymbol{x})$ are $\lambda = 0, \pm R$ with $R^2 = X^2 + Y^2 + Z^2$, so that degeneracy occurs if and only if $R = 0$, which provides degeneracy points,

$$\begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{if and only if} \quad \frac{a}{2} = -\frac{b}{5}. \tag{258}$$

$$\begin{pmatrix} 0 \\ \pm\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} \pm\frac{1}{\sqrt{2}} \\ 0 \\ \pm\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} \pm\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad \text{if and only if} \quad \frac{a}{1} = \frac{b}{10}. \tag{259}$$

$$\begin{pmatrix} \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \end{pmatrix}, \quad \text{if and only if} \quad \frac{a}{4} = \frac{b}{15}. \tag{260}$$

Though this Hamiltonian is a three-level model, the eigenvalue problem is easy to solve because of the existence of zero eigenvalue. The Chern numbers of respective eigen-line bundles can be calculated in a manner similar to that used in Sec. 5. Without detail calculation for Chern numbers, we here give a result on Chern numbers.

Proposition 12.1 The parameter space $\mathbb{R}^2 - \{0\}$ for the O-invariant Hamiltonian (256) reduces to the unit circle. In association with the positive eigenvalue $\lambda^+ = R$, an eigen-line bundle is determined on each arc between consecutive degeneracy points on the unit circle. The Chern numbers c^+ assigned to respective iso-Chern domains (or arcs) are shown in Fig. 9.

In what follows, we give a sketch of calculation for Chern numbers of the eigen-line bundle associated with the positive eigenvalue λ^+, along the tangent line $(a, b) = (1, \varepsilon)$ to

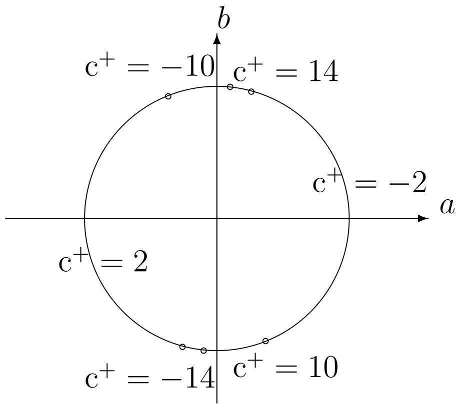

Figure 9: Chern numbers c^+ of the eigen-line bundle associated with the positive eigenvalue are assigned to arcs separated by degeneracy points.

the unit circle at $(a, b) = (1, 0)$. The Chern number contribution from exceptional points for the "up" eigenvector is given in the following table;

ε		$-\frac{5}{2}$		$\frac{5}{3}$		$\frac{15}{4}$		10	
$\boldsymbol{n}_{\pm}^{(+)}$	$+$	$\deg(C_4)$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\boldsymbol{a}_{\pm j}^{(+)}$	$+$	$\deg(C_4)$	(empty)	(empty)	$+$	$+$	$+$	$\deg(C_2)$	$-$
$\boldsymbol{b}_{\pm j}^{(+)}$	$+$	$\deg(C_4)$	(empty)	(empty)	$+$	$+$	$+$	$\deg(C_2)$	$-$
$\boldsymbol{c}_{\pm k}^{(+)}$	(empty)	(empty)	(empty)	(empty)	$-$	$\deg(C_3)$	$+$	$\deg(C_2)$	(empty)
Ch.no	10	non-def	-2	-2	-2	non-def	14	non-def	-10

Here, the symbols $\boldsymbol{n}_{\pm}^{(+)}$ *etc.* in the leftmost column stand for exceptional points, which we do not describe explicitly. The superscript $(+)$ indicates that those exceptional points are assigned to the "up" eigenvector. The subscripts j and k of $\boldsymbol{a}_{\pm j}^{(+)}$, $\boldsymbol{b}_{\pm j}^{(+)}$, $\boldsymbol{c}_{\pm k}^{(+)}$ range from 1 to 2 and from 1 to 4, respectively. The signs $\pm$ in the subscript of $\boldsymbol{a}_{\pm j}^{(+)}$, *etc*, indicate the sign of the z-component of each exceptional point, and hence $\boldsymbol{a}_{\pm j}^{(+)}$ denote four points, *etc.* The signs $\pm$ in the table stand for the Chern number contribution from the exceptional points listed in the leftmost column of the table, where $+$ (resp. $-$) means that $+1$ (resp. -1) is assigned to each exceptional point. Although the signs $\pm$ are alloted in the columns below $\varepsilon = \frac{15}{4}$ and $\varepsilon = 10$, which seems to assign a Chern number contribution from exceptional points, no Chern number is defined at $\varepsilon = \frac{1}{4}$ and $\varepsilon = 10$ because of the existence of degeneracy points.

We observe from this table that there are special values of the parameter ε at which exceptional points changes their character.

(i) At $\varepsilon = -\frac{5}{2}$, the exceptional points $\boldsymbol{n}_{\pm}^{(+)}$ become degeneracy points and then return to exceptional points of different character for $\varepsilon > -\frac{5}{2}$, and further the exceptional points $\boldsymbol{a}_{\pm j}^{(+)}$ and $\boldsymbol{b}_{\pm j}^{(+)}$ change into degeneracy points, forming an O-orbit of type C_4 together with $\boldsymbol{n}_{\pm}^{(+)}$, and then vanish for $\varepsilon > -\frac{5}{2}$ (see Fig. 10);

$$\boldsymbol{n}_{\pm}^{(+)} = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}, \quad \boldsymbol{a}_{\pm j}^{(+)} \to \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{b}_{\pm j}^{(+)} \to \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}, \ j = 1, 2. \tag{261}$$

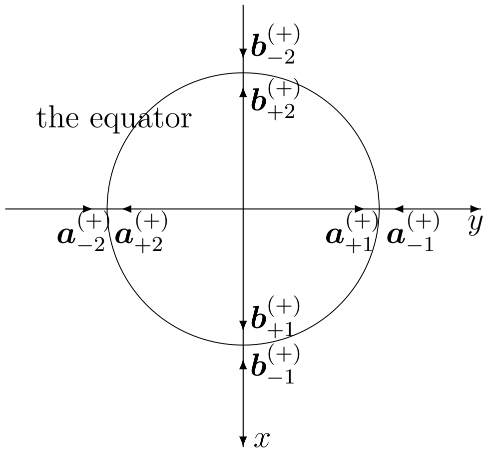

Figure 10: As $\varepsilon \to -\frac{5}{2}$, the exceptional points $\boldsymbol{a}_{\pm j}^{(+)}$ and $\boldsymbol{b}_{\pm j}^{(+)}$ are getting together to be degeneracy points and then to vanish pairwise for $\varepsilon > -\frac{5}{2}$.

(ii) When the parameter ε passes the value $\varepsilon = \frac{5}{3}$, the exceptional points $\boldsymbol{a}_{\pm j}^{(+)}$, $\boldsymbol{b}_{\pm j}^{(+)}$, and $\boldsymbol{c}_{\pm k}^{(+)}$ branch off from the north and south poles (see Fig. 11).

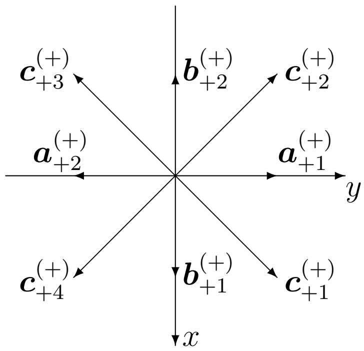

Figure 11: When the parameter value passes $\varepsilon = \frac{5}{3}$ upward, new exceptional points $\boldsymbol{a}_{+j}^{(+)}$, $\boldsymbol{b}_{+j}^{(+)}$, and $\boldsymbol{c}_{+k}^{(+)}$ branch off from the north pole $\boldsymbol{n}_{+}^{(+)}$.

(iii) The exceptional points $\boldsymbol{n}_{\pm}^{(+)}$, $\boldsymbol{a}_{\pm j}^{(+)}$, and $\boldsymbol{b}_{\pm j}^{(+)}$ remains to be exceptional at $\varepsilon = \frac{15}{4}$, but the exceptional points $\boldsymbol{c}_{\pm k}^{(+)}$ become degeneracy points at $\varepsilon = \frac{15}{4}$, forming an O-orbit of type C_3;

$$\boldsymbol{a}_{\pm j}^{(+)} = \begin{pmatrix} 0 \\ \pm\frac{1}{\sqrt{3}} \\ \pm\sqrt{\frac{2}{3}} \end{pmatrix}, \quad \boldsymbol{b}_{\pm j}^{(+)} = \begin{pmatrix} \pm\frac{1}{\sqrt{3}} \\ 0 \\ \pm\sqrt{\frac{2}{3}} \end{pmatrix} \quad \boldsymbol{c}_{\pm k}^{(+)} = \begin{pmatrix} \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \end{pmatrix} \quad \text{for} \quad \varepsilon = \frac{15}{4}. \tag{262}$$

(iv) The exceptional points $\boldsymbol{a}^{(+)}_{\pm j}$, $\boldsymbol{b}^{(+)}_{\pm j}$, and $\boldsymbol{c}^{(+)}_{\pm k}$ become degeneracy points at $\varepsilon = 10$, forming an O-orbit of type C_2, but the $\boldsymbol{c}^{(+)}_{\pm k}$ vanish for $\varepsilon > 10$ (see Fig. 12);

$$\boldsymbol{a}^{(+)}_{\pm j} \to \begin{pmatrix} 0 \\ \pm\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \boldsymbol{b}^{(+)}_{\pm j} \to \begin{pmatrix} \pm\frac{1}{\sqrt{2}} \\ 0 \\ \pm\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \boldsymbol{c}^{(+)}_{\pm k} \to \begin{pmatrix} \pm\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}. \tag{263}$$

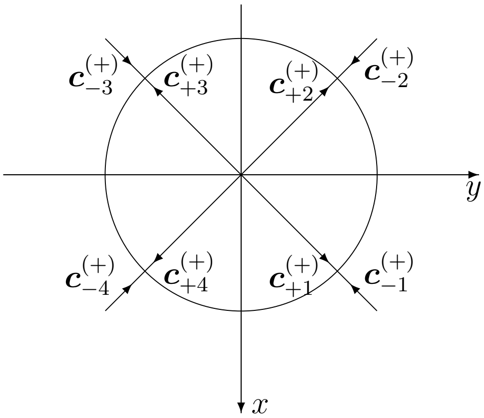

Figure 12: As $\varepsilon \to 10$, the exceptional points $\boldsymbol{c}^{(+)}_{\pm k}$ are getting together pairwise, and vanish for $\varepsilon > 10$.

Since the Chern number of the eigen-line bundle associated with the negative eigenvalue λ^- is minus that of the eigen-line bundle associated with the positive eigenvalue λ^+, and the Chern number of the eigen-line bundle associated with the zero eigenvalue is always zero, the Chern number variation along the circle $(a, b) = (\cos\theta, \sin\theta)$ is now summarized in Fig. 13.

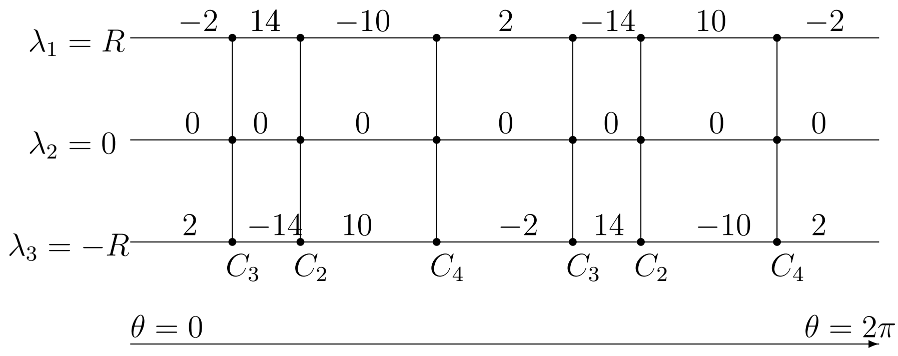

Figure 13: Chern number variation for the Hamiltonian (256)

Fig. 13 shows that the amount of the change in Chern numbers which is observed when the control parameter passes each degeneracy point is twice the order of the orbit,

$2\#(O/C_k)$, $k = 2, 3, 4$, or twice the number of degeneracy points, up to the sign. If the number $\#(O/C_k)$ is assigned to a degeneracy point of multiplicity two, twice the number $\#(O/C_k)$ is assigned to the triple degeneracy point up to sign. A possible explanation of this fact is that since the triple crossing of the energy levels is considered as the simultaneous occurrence of three double crossings of the levels, the total delta-Chern is evaluated as the sum of contributions from all crossings:

$$\begin{pmatrix} +\Delta c \\ -\Delta c \\ 0 \end{pmatrix} + \begin{pmatrix} +\Delta c \\ 0 \\ -\Delta c \end{pmatrix} + \begin{pmatrix} 0 \\ +\Delta c \\ -\Delta c \end{pmatrix} = \begin{pmatrix} +2\Delta c \\ 0 \\ -2\Delta c \end{pmatrix}, \tag{264}$$

where $|\Delta c| = \#(O/C_k)$.

13 A relation to the Berry phase

So far we have mainly treated 3×3 Hermitian matrices depending on the control parameters (a, b) and on the variable $\boldsymbol{x} \in S^2$. The dimension of the total parameter space $\mathbb{R}^2 \times S^2$ is four. We note here that the distinction between the dynamical variables and the control parameters is made by the symmetry group action; the symmetry group acts on the dynamical variables but not on the control parameters.

If we don't care about the distinction between the control parameters and the physical variables, nor about the topology of the set of physical variables, we are allowed to think of the Hamiltonian as depending on parameters in $\mathbb{R}^4$. We now suppose that the Hamiltonian is not restricted to two- or three-level model ones but it has doubly degenerate eigenvalues which are so far from the other eigenvalues that the eigenspace associated with degenerate eigenvalues can be treated separately. Since the codimension for the doubly degenerate eigenvalues is three for Hermitian matrices, there is a degeneracy curve in the parameter space $\mathbb{R}^4$, which we denote by $\boldsymbol{c}(s) \in \mathbb{R}^4$. Further, we denote by $|e_k(\boldsymbol{c}(s))\rangle$ the orthonormalized eigenvectors associated with degenerate eigenvalues $\lambda_k(\boldsymbol{c}(s))$, $k = 1, 2$. Then, we have

$$\lambda_1(\boldsymbol{c}(s)) = \lambda_2(\boldsymbol{c}(s)). \tag{265}$$

and

$$H(\boldsymbol{c}(s))|e_k(\boldsymbol{c}(s))\rangle = \lambda_k(\boldsymbol{c}(s))|e_k(\boldsymbol{c}(s))\rangle, \quad \langle e_k(\boldsymbol{c}(s))|e_j(\boldsymbol{c}(s))\rangle = \delta_{kj}, \tag{266}$$

where the symbol $\langle\cdot|\cdot\rangle$ stands for the inner product on the Hilbert space concerned.

We assume that $|e_k(\boldsymbol{c}(s))\rangle$ and $\lambda_k(\boldsymbol{c}(s))$ are smooth in s. Differentiating the eigenvalue equation with respect to the parameter s, we obtain

$$\frac{dH(\boldsymbol{c}(s))}{ds}|e_k(\boldsymbol{c}(s))\rangle + H(\boldsymbol{x}(s))\frac{d|e_k(\boldsymbol{c}(s))\rangle}{ds} = \frac{d\lambda_k(\boldsymbol{c}(s))}{ds}|e_k(\boldsymbol{c}(s))\rangle + \lambda_k(\boldsymbol{c}(s))\frac{d|e_k(\boldsymbol{c}(s))\rangle}{ds}. \tag{267}$$

Taking the inner product of the above equation and the eigenvector $|e_j(\boldsymbol{c}(s)\rangle$, and using the fact that $\lambda_1 = \lambda_2$, we obtain

$$\left\langle e_j(\boldsymbol{c}(s)) \left| \frac{dH(\boldsymbol{c}(s))}{ds} \right| e_k((s)) \right\rangle = \frac{d\lambda_k(\boldsymbol{c}(s))}{ds}\delta_{jk}. \tag{268}$$

Let us denote by $\boldsymbol{\nabla} H$ the matrix each of whose entries is the gradient of the entry of H concerned; $\boldsymbol{\nabla} H = (\boldsymbol{\nabla} H_{\ell m})$ with $H = (H_{\ell m})$. Then, one has

$$\left\langle e_j(\boldsymbol{c}(s)) \left| \frac{dH(\boldsymbol{c}(s))}{ds} \right| e_k(\boldsymbol{c}(s)) \right\rangle = \langle e_j(\boldsymbol{c}(s)) | \boldsymbol{\nabla} H(\boldsymbol{c}(s)) | e_k(\boldsymbol{c}(s)) \rangle \cdot \frac{d\boldsymbol{c}(s)}{ds}, \tag{269}$$

where the center dot stands for the inner product on $\mathbb{C}^4$, which is formally extended from that on $\mathbb{R}^4$. In particular, for (j,k) with $j \neq k$, we have, from the above equations,

$$\langle e_j(\boldsymbol{c}(s)) | \boldsymbol{\nabla} H(\boldsymbol{c}(s)) | e_k(\boldsymbol{c}(s)) \rangle \cdot \frac{d\boldsymbol{c}(s)}{ds} = 0, \tag{270}$$

and for (j,k) with $j = k$,

$$\begin{aligned} &\langle e_1(\boldsymbol{c}(s)) | \boldsymbol{\nabla} H(\boldsymbol{c}(s)) | e_1(\boldsymbol{c}(s)) \rangle \cdot \frac{d\boldsymbol{c}(s)}{ds} - \langle e_2(\boldsymbol{c}(s)) | \boldsymbol{\nabla} H(\boldsymbol{c}(s)) | e_2(\boldsymbol{c}(s)) \rangle \cdot \frac{d\boldsymbol{c}(s)}{ds} \\ = &\ \frac{d\lambda_1(\boldsymbol{c}(s))}{ds} - \frac{d\lambda_2(\boldsymbol{c}(s))}{ds} \ = \ 0. \end{aligned} \tag{271}$$

This implies that the vectors

$$\Re\langle e_j(\boldsymbol{c}(s)) | \boldsymbol{\nabla} H(\boldsymbol{c}(s)) | e_k(\boldsymbol{c}(s)) \rangle, \quad \Im\langle e_j(\boldsymbol{c}(s)) | \boldsymbol{\nabla} H(\boldsymbol{c}(s)) | e_k(\boldsymbol{c}(s)) \rangle, \tag{272a}$$

$$\frac{1}{2}\Big(\langle e_1(\boldsymbol{c}(s)) | \boldsymbol{\nabla} H(\boldsymbol{c}(s)) | e_1(\boldsymbol{c}(s)) \rangle - \langle e_2(\boldsymbol{c}(s)) | \boldsymbol{\nabla} H(\boldsymbol{c}(s)) | e_2(\boldsymbol{c}(s)) \rangle \Big) \tag{272b}$$

are orthogonal to the tangent vector $\dfrac{d\boldsymbol{c}(s)}{ds}$ to the curve $\boldsymbol{c}(s)$, where $(j,k) = (1,2)$, $(2,1)$.

In order to consider the Hamiltonian evaluated at points different from the degeneracy curve $\boldsymbol{c}(s)$, we fix the parameter s at s_0 for the time being and set $\boldsymbol{c}(s_0) = \boldsymbol{c}_0$. We further set

$$a\boldsymbol{\xi} = \frac{1}{2}\Big(\langle e_1(\boldsymbol{c}_0) | \boldsymbol{\nabla} H(\boldsymbol{c}_0) | e_1(\boldsymbol{c}_0) \rangle - \langle e_2(\boldsymbol{c}_0) | \boldsymbol{\nabla} H(\boldsymbol{c}_0) | e_2(\boldsymbol{c}_0) \rangle \Big), \tag{273a}$$

$$b\boldsymbol{\eta} = \langle e_1(\boldsymbol{c}_0) | \boldsymbol{\nabla} H(\boldsymbol{c}_0) | e_2(\boldsymbol{c}_0) \rangle, \tag{273b}$$

where a and b are introduced so as to make $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ have the unit length. We here suppose that the $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and $\overline{\boldsymbol{\eta}}$ span the plane Π_0 orthogonal to the curve $\boldsymbol{c}(s)$ at $\boldsymbol{c}(s_0) = \boldsymbol{c}_0$ on the identification $\Pi_0 \cong \mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C}$. Let $\boldsymbol{\zeta}$ be a vector sitting in the plane Π_0 and ε denote an infinitesimal parameter. Then, on setting $\boldsymbol{r} = \boldsymbol{c}_0 + \varepsilon\boldsymbol{\zeta}$, we obtain, by differentiation and arrangement,

$$\frac{1}{2}\Big(\langle e_1(\boldsymbol{c}_0) | H(\boldsymbol{r}) | e_1(\boldsymbol{c}_0) \rangle - \langle e_2(\boldsymbol{c}_0) | H(\boldsymbol{r}) | e_2(\boldsymbol{c}_0) \rangle \Big) = \varepsilon a \boldsymbol{\xi} \cdot \boldsymbol{\zeta}, \tag{274a}$$

$$\langle e_1(\boldsymbol{c}_0) | H(\boldsymbol{r}) | e_2(\boldsymbol{c}_0) \rangle = \varepsilon b \boldsymbol{\eta} \cdot \boldsymbol{\zeta}. \tag{274b}$$

Further, setting

$$\mu(\boldsymbol{r}) = \frac{1}{2}\Big(\langle e_1(\boldsymbol{c}_0) | H(\boldsymbol{r}) | e_1(\boldsymbol{c}_0) \rangle + \langle e_2(\boldsymbol{c}_0) | H(\boldsymbol{r}) | e_2(\boldsymbol{c}_0) \rangle \Big), \tag{275}$$

we have

$$\mu(\boldsymbol{r}) = \mu(\boldsymbol{c}_0) + \varepsilon\nabla\mu(\boldsymbol{x}_0)\cdot\boldsymbol{\zeta}. \tag{276}$$

We are now in a position to express the Hamiltonian $H(\boldsymbol{r})$ evaluated approximately on the plane Π_0. From (274) and (276), it follows that

$$\begin{aligned}
\langle e_1(\boldsymbol{c}_0)|H(\boldsymbol{r})|e_1(\boldsymbol{c}_0)\rangle &= \mu(\boldsymbol{c}_0) + \varepsilon\nabla\mu(\boldsymbol{c}_0)\cdot\boldsymbol{\zeta} + \varepsilon a\boldsymbol{\xi}\cdot\boldsymbol{\zeta}, && (277a)\\
\langle e_2(\boldsymbol{c}_0)|H(\boldsymbol{r})|e_2(\boldsymbol{c}_0)\rangle &= \mu(\boldsymbol{c}_0) + \varepsilon\nabla\mu(\boldsymbol{c}_0)\cdot\boldsymbol{\zeta} - \varepsilon a\boldsymbol{\xi}\cdot\boldsymbol{\zeta}, && (277b)\\
\langle e_1(\boldsymbol{c}_0)|H(\boldsymbol{r})|e_2(\boldsymbol{c}_0)\rangle &= \varepsilon b\boldsymbol{\eta}\cdot\boldsymbol{\zeta}. && (277c)
\end{aligned}$$

Then the Hamiltonian evaluated at $\boldsymbol{r} = \boldsymbol{c}_0 + \varepsilon\boldsymbol{\zeta} \in \Pi_0$ and restricted to the eigenspace spanned by the eigenvectors $|e_k(\boldsymbol{c}_0)\rangle$, $k = 1, 2$, is expressed as

$$H(\boldsymbol{r}) = \big(\mu(\boldsymbol{c}_0) + \varepsilon\nabla\mu(\boldsymbol{c}_0)\cdot\boldsymbol{\zeta}\big)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \varepsilon\begin{pmatrix}a\boldsymbol{\xi}\cdot\boldsymbol{\zeta} & b\boldsymbol{\eta}\cdot\boldsymbol{\zeta}\\ b\overline{\boldsymbol{\eta}\cdot\boldsymbol{\zeta}} & -a\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\end{pmatrix}. \tag{278}$$

If we consider that the plane Π_0 carries the dynamical variables only, we may introduce the coordinates (q_0, q_1, q_2) on the plane Π_0 by setting

$$q_0 = a\boldsymbol{\xi}\cdot\boldsymbol{\zeta}, \quad q_1 - iq_2 = b\boldsymbol{\eta}\cdot\boldsymbol{\zeta}, \tag{279}$$

where we have paid little attention to the orientation of the coordinate system. Then, from the second term of the right-hand side of (278), we obtain a local Hamiltonian with vanishing trace in the form

$$\begin{pmatrix} q_0 & q_1 - iq_2\\ q_1 + iq_2 & -q_0\end{pmatrix}. \tag{280}$$

The eigenvectors of this Hamiltonian are defined only locally on the tangent plane Π_0, and the locally defined eigenvectors, which we have called "up" and "down" eigenvectors so far, are put together to form an eigen-line bundle. This Hamiltonian has been extensively discussed since Berry [21, 22].

If we consider that the plane Π_0 carries one control parameter and two dynamical variables, we may denote them by (t, q_1, q_2) in place of (q_0, q_1, q_2), where we suppose that an isotropy subgroup of the symmetry group acts on (q_1, q_2). However, as the two-dimensional subspace for the dynamical variables (q_1, q_2) don't need to be the same as those given in (280), the local Hamiltonian with vanishing trace may take the form of (83), after a linear transformation of (t, q_1, q_2).

In a typical case, we have the local Hamiltonian, in place of (280),

$$H(t, q) = \begin{pmatrix} t & q_1 - iq_2\\ q_1 + iq_2 & -t\end{pmatrix}, \tag{281}$$

for which the evolution of eigenvalues against t is depicted in Fig. 4. We now discuss this Hamiltonian from the view point of Berry phase. The eigenvalues are given by

$$\lambda^{\pm} = \pm\sqrt{t^2 + |q|^2}, \quad |q|^2 = q_1^2 + q_2^2, \tag{282}$$

The "up" eigenvector associated with λ^+ is expressed as

$$|u^+_{\rm up}(t, q)\rangle = \frac{1}{N_{\rm up}}\begin{pmatrix} q_1 - iq_2\\ \sqrt{t^2 + |q|^2} - t\end{pmatrix}, \quad N_{\rm up} = \sqrt{2\big(t^2 + |q|^2 - t\sqrt{t^2 + |q|^2}\big)}. \tag{283}$$

The exceptional point is the origin $q = 0$, which is assigned to $|u_{\rm up}^{+}(t,q)\rangle$ for $t > 0$ only. We now consider the parallel translation of the vector

$$|v(\tau)\rangle = e^{i\theta(\tau)}|u_{\rm up}^{+}(t,q(\tau))\rangle \tag{284}$$

along a circle $q_1 + iq_2 = \rho e^{i\tau}$ with $\rho > 0$, where t is fixed. Let

$$P = |u_{\rm up}^{+}(t,q)\rangle\langle u_{\rm up}^{+}(t,q)| = |u_{\rm down}^{+}(t,q)\rangle\langle u_{\rm down}^{+}(t,q)|, \quad q \neq 0, \tag{285}$$

be the projection operator. Then, the vector $|v(\tau)\rangle$ parallel translates along the curve $q_1 + iq_2 = \rho e^{i\tau}$, if and only if

$$P\frac{d}{d\tau}|v(\tau)\rangle = |u_{\rm up}^{+}(t,q(\tau))\rangle ie^{i\theta}\Big(\frac{d\theta}{dt} - \frac{\rho^2}{N_{\rm up}^2}\Big) = 0, \quad \rho = |q|. \tag{286}$$

Hence, after completing the parallel translation along the closed curve $q_1+iq_2 = \rho e^{i\tau}$ with $0 \leq \tau \leq 2\pi$, the phase of the section $|v(\tau)\rangle$ of the eigen-line bundle associated with λ^+ has increased by the angle

$$\frac{2\pi\rho^2}{N_{\rm up}^2}, \quad |q| = \rho > 0, \tag{287}$$

which is viewed as a Berry phase.

If the parameter space is $\mathbb{R}^m$ with $m \geq 4$, the curve $\boldsymbol{c}(s)$ is viewed as a curve sitting in the degeneracy surface (or submanifold) determined by the degeneracy condition. The variation vector $\boldsymbol{r} = \boldsymbol{c}_0+\varepsilon\boldsymbol{\zeta}$ sitting in the plane Π_0 orthogonal to the degeneracy curve $\boldsymbol{c}(s)$ at $\boldsymbol{c}_0$ has components some of which are tangent to the degeneracy surface and the others normal to the degeneracy surface. In order to observe transition states accompanying the crossing of the boundary (or degeneracy surface), we choose to use the coordinates associated with the normal components, in forming a linearized Hamiltonian from (278). Then, like (280), we obtain a linear Hamiltonian described in terms of the local coordinates q_j associated with transverse directions.

14 Possible extensions of delta-Chern analysis

So far we have encountered some cases where the linearization method fails (see Sec. 8.3). If a linearization method fails at a degeneracy point of S^2, we have to reform the linearization method, for example, by taking quadratic terms in the expansion the Hamiltonian at a degeneracy point in terms of local coordinates. Another way of extension is to find a way to the delta-Chern formula assigned to a degeneracy point at which eigenvalues are triply degenerate. This section is devoted to the discussion of these extensions.

14.1 Hessian of the Hamiltonian and symmetry

We now assume that the first derivative of the Hamiltonian vanishes at a point $\boldsymbol{x}_0$. Put in detail, for a positively oriented frame $\boldsymbol{\xi}_k$ on the tangent plane to S^2 at $\boldsymbol{x}_0$, the derivatives

$\boldsymbol{\xi}_k \cdot \nabla H(\boldsymbol{x}_0)$ vanish. In order to take into account the second derivatives of the Hamiltonian, we need to calculate the Hessian of the Hamiltonian with respect to the frame $\boldsymbol{\xi}_k$. To this end, one needs covariant derivatives of functions on the sphere.

We now start with a geodesic $\boldsymbol{x}(s)$ passing $\boldsymbol{x}_0$ at $s = 0$, where s is the arc length parameter. The second derivative of a function $f(\boldsymbol{x})$ with respect to s along $\boldsymbol{x}(s)$ is evaluated at $s = 0$ to provide the definition of the Hessian of f at $\boldsymbol{x}_0$ in the form

$$\frac{d^2}{ds^2} f(\boldsymbol{x}(s))\bigg|_{s=0} = \boldsymbol{\zeta} \cdot (\mathrm{Hess} f(\boldsymbol{x}_0))\boldsymbol{\zeta}, \quad \boldsymbol{\zeta} = \frac{d\boldsymbol{x}}{ds}(0), \tag{288}$$

where $\mathrm{Hess} f(\boldsymbol{x}_0)$ is viewed as a linear transformation of the tangent space $T_{\boldsymbol{x}_0}(S^2)$. The second derivative of the function $f(\boldsymbol{x})$ with respect to s is written out as

$$\frac{d^2}{ds^2} f(\boldsymbol{x}(s)) = \frac{d^2\boldsymbol{x}}{ds^2} \cdot \nabla f(\boldsymbol{x}) + \frac{d\boldsymbol{x}}{ds} \cdot \nabla^{2\otimes} f(\boldsymbol{x}) \frac{d\boldsymbol{x}}{ds}, \tag{289}$$

where $\nabla^{2\otimes}$ denotes the operator matrix given by

$$\nabla^{2\otimes} = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x \partial z} \\ \frac{\partial^2}{\partial y \partial x} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial y \partial z} \\ \frac{\partial^2}{\partial z \partial x} & \frac{\partial^2}{\partial z \partial y} & \frac{\partial^2}{\partial z^2} \end{pmatrix}. \tag{290}$$

Taking into account the geodesic equation on S^2

$$\frac{d^2\boldsymbol{x}}{ds^2} + \boldsymbol{x} = 0, \tag{291}$$

we find from (288) and (289) that the Hessian of $f(\boldsymbol{x})$ at $\boldsymbol{x}_0$ is determined through

$$\boldsymbol{\zeta} \cdot (\mathrm{Hess} f(\boldsymbol{x}_0))\boldsymbol{\zeta} = -\boldsymbol{x}_0 \cdot \nabla f(\boldsymbol{x}_0) + \boldsymbol{\zeta} \cdot \nabla^{2\otimes} f(\boldsymbol{x}_0)\boldsymbol{\zeta}, \tag{292}$$

where $\boldsymbol{\zeta}$ is a tangent vector in $T_{\boldsymbol{x}_0}(S^2)$. The Hessian in generic form is defined through

$$\begin{aligned} &\boldsymbol{\eta} \cdot \mathrm{Hess} f(\boldsymbol{x}_0)(\boldsymbol{\zeta}) \\ =&\frac{1}{2}\Big((\boldsymbol{\eta} + \boldsymbol{\zeta}) \cdot \mathrm{Hess} f(\boldsymbol{x}_0)(\boldsymbol{\eta} + \boldsymbol{\zeta}) - \boldsymbol{\eta} \cdot \mathrm{Hess} f(\boldsymbol{x}_0)(\boldsymbol{\eta}) - \boldsymbol{\zeta} \cdot \mathrm{Hess} f(\boldsymbol{x}_0)(\boldsymbol{\zeta})\Big). \end{aligned} \tag{293}$$

The Hessian of the Hamiltonian, $\mathrm{Hess}\, H(\boldsymbol{x}_0)$, at $\boldsymbol{x}_0$ is now defined in a similar manner by applying the above procedure to each components of the Hamiltonian and takes values in $\mathrm{End}(T_{\boldsymbol{x}_0}(S^2)) \otimes \mathbb{C}^{n\times n}$, where $\mathrm{End}(T_{\boldsymbol{x}_0}(S^2))$ indicates that each component $\mathrm{Hess}\, h_{\ell m}(\boldsymbol{x}_0)$ of $\mathrm{Hess} H(\boldsymbol{x}_0)$ is viewed as a linear map of the tangent space $T_{\boldsymbol{x}_0}(S^2)$. The second-order approximation of the full Hamiltonian at $\boldsymbol{x}_0$ is given by

$$H_2(q; \boldsymbol{x}_0) := \frac{1}{2}\sum_{j,k} q_j q_k \boldsymbol{\xi}_j \cdot (\mathrm{Hess}\, H(\boldsymbol{x}_0))\boldsymbol{\xi}_k = \frac{1}{2}\left(\sum_{j,k} q_j q_k \boldsymbol{\xi}_j \cdot (\mathrm{Hess}\, h_{\ell m}(\boldsymbol{x}_0))\boldsymbol{\xi}_k\right). \tag{294}$$

Like (8), for $g \in G \subset SO(3)$, the Hamiltonian $H_2(q; \boldsymbol{x}_0)$ is shown to be subject to the transformation

$$H_2(q; g\boldsymbol{x}_0) = \mathrm{Ad}_{D(g)} H_2(q; \boldsymbol{x}_0), \quad g \in G, \tag{295}$$

if the frame $g\boldsymbol{\xi}_k$ is adopted at $g\boldsymbol{x}_0$. The proof is easy to perform. Since $g \in G$ is an isometry of the sphere, if $\boldsymbol{x}(s)$ is a geodesic, so is $g\boldsymbol{x}(s)$. Then, the symmetry condition $H(g\boldsymbol{x}(s)) = \mathrm{Ad}_{D(g)}H(\boldsymbol{x}(s))$ is twice differentiated with respect to s at $s = 0$ to yield the above equation. For $h \in G_0$, Eq. (295) becomes

$$\mathrm{Ad}_{D(h)}H_2(q;\boldsymbol{x}_0) = H_2(h^{(2)}q;\boldsymbol{x}_0), \quad h \in G_0, \tag{296}$$

where $h\boldsymbol{\xi}_k = \sum_j h^{(2)}_{jk}\boldsymbol{\xi}_j$.

As in Sec. 6.1, we here consider a one-parameter Hamiltonian $H(c(t),\boldsymbol{x})$, but we assume that the linear approximation fails because of $\boldsymbol{\xi}_k \cdot \nabla H(c(0),\boldsymbol{x}_0) = 0$. However, if $\mathrm{Hess}\,H(c(0),\boldsymbol{x}_0)$ does not vanish, we can define the local Hamiltonian, in place of (52), to be

$$H_{\mathrm{loc}}(t,q;\boldsymbol{x}_0) = H(c(0),\boldsymbol{x}_0) + t\dot{H}(c(0),\boldsymbol{x}_0) + \frac{1}{2}\sum_k q_j q_k \boldsymbol{\xi}_j \cdot (\mathrm{Hess}\,H(c(0),\boldsymbol{x}_0))\boldsymbol{\xi}_k. \tag{297}$$

Further, we can put the Hamiltonian $H_{\mathrm{loc}}(t,q;\boldsymbol{x}_0)$ in normal form by taking a basis with respect to which the isotropy subgroup at $\boldsymbol{x}_0$ is represented in a diagonal matrix form. We denote the representation matrix by $\tilde{D}(g)$ and the local Hamiltonian in normal form by the same symbol $K_{\mathrm{loc}}(t,q;\boldsymbol{x}_0)$ as that in the case of linear approximation. Like (296), the local Hamiltonian $K_{\mathrm{loc}}(t,q;\boldsymbol{x}_0)$ transforms according to

$$\mathrm{Ad}_{\tilde{D}(h)}K_{\mathrm{loc}}(t,q;\boldsymbol{x}_0) = K_{\mathrm{loc}}(t,h^{(2)}q;\boldsymbol{x}_0), \quad h \in G_0. \tag{298}$$

If we adopt the basis $|e_j(\boldsymbol{x}_0')\rangle := \tilde{D}(g)|e_j(\boldsymbol{x}_0)\rangle$ and the frame $\boldsymbol{\xi}_k' = g\boldsymbol{\xi}_k$ at $\boldsymbol{x}_0' = g\boldsymbol{x}_0$, the local Hamiltonian at $\boldsymbol{x}_0'$ takes the same form as that at $\boldsymbol{x}_0$, like (92),

$$K_{\mathrm{loc}}'(t,q;\boldsymbol{x}_0') = K_{\mathrm{loc}}(t,q;\boldsymbol{x}_0). \tag{299}$$

This is a core equation in the delta-Chern analysis. Starting with this equation, we can modify the reasoning which leads to the global delta-Chern formula in the case of linear local Hamiltonians, except for $\Delta W = \mathrm{sgn}(\det C(K))$, in order to make it applicable in the case of, say, quadratic local Hamiltonians. This is because the winding number is independent of the choice of the frame at a degeneracy point and of the basis with respect to which the local Hamiltonian is expressed, because the homotopic deformation of the transition function is valid as well, and because $\Delta^{\pm}c(g\boldsymbol{x}_0)$ is constant on the orbit $\mathcal{O}_{\boldsymbol{x}_0}$. It then turns out that if we can evaluate ΔW directly, we can obtain the delta-Chern formula, like Theorem 9.1.

14.2 Delta-Chern arising from second-order terms

We here give a simple but expressive example for a case study, in which the linear approximation of the Hamiltonian fails but the second-order approximation works. We consider the Hamiltonian

$$\begin{aligned} H_\alpha(\boldsymbol{x}) =& (1-\alpha)\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} + \alpha \begin{pmatrix} z & (x-iy)^2 \\ (x+iy)^2 & -z \end{pmatrix} \\ =& \begin{pmatrix} 1-\alpha+\alpha z & \alpha(x-iy)^2 \\ \alpha(x+iy)^2 & -1+\alpha-\alpha z \end{pmatrix}, \quad 0 \le \alpha \le 1, \quad \boldsymbol{x} \in S^2. \end{aligned} \tag{300}$$

The symmetry group $SO(2)$ given by

$$h(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix} \tag{301}$$

acts on S^2 and is represented as

$$D(h(\theta)) = \begin{pmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{pmatrix}. \tag{302}$$

The Hamiltonian (300) is invariant under the $SO(2)$ action,

$$D(h(\theta))H_\alpha(\boldsymbol{x})D(h(\theta))^{-1} = H_\alpha(h(\theta)\boldsymbol{x}). \tag{303}$$

The eigenvalues of $H_\alpha(\boldsymbol{x})$ are easily found to be

$$\mu^\pm = \pm\sqrt{(1-\alpha+\alpha z)^2 + \alpha^2(x^2+y^2)^2}. \tag{304}$$

Hence, degeneracy in eigenvalues occurs if and only if

$$1-\alpha+\alpha z = 0, \quad \alpha(x^2+y^2) = 0. \tag{305}$$

The unique solution to this is given by

$$\boldsymbol{x}_0 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad \alpha = \frac{1}{2}. \tag{306}$$

The isotropy subgroup at $\boldsymbol{x}_0$ is $SO(2)$ itself. We take at $\boldsymbol{x}_0$ the frame

$$\boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\xi}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \tag{307}$$

which is positively oriented in the sense that $\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 = \boldsymbol{x}_0$. The Cartesian coordinates are introduced through $\sum q_k \boldsymbol{\xi}_k$ on the tangent plane Π_0 to S^2 at $\boldsymbol{x}_0$. With respect to the frame $\boldsymbol{\xi}_k$, the isotropy subgroup is represented as

$$h^{(2)}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{308}$$

As is easily seen from the Hamiltonian (300), the first derivative of $H_\alpha(\boldsymbol{x}_0)$ vanishes on the tangent plane at $\boldsymbol{x}_0$; $\nabla H_\alpha(\boldsymbol{x}_0)\cdot\boldsymbol{\xi}_k = 0$. However, the derivative of H with respect to α does not vanish and is evaluated as

$$\dot{H}_\alpha(\boldsymbol{x}_0) = \begin{pmatrix} -2 & \\ & 2 \end{pmatrix}. \tag{309}$$

If we adopt the local Hamiltonian $H_{\frac{1}{2}}(\boldsymbol{x}_0) + t\dot{H}_{\frac{1}{2}}(\boldsymbol{x}_0) + \sum q_k \nabla H_{\frac{1}{2}}(\boldsymbol{x}_0) \cdot \boldsymbol{\xi}_k$, the relevant matrix $C(K)$ is of rank one and the linearization method fails. We here note that the present local Hamiltonian is in normal form, since the isotropy subgroup is represented in the diagonal form (302).

We apply the formulas (292) and (293) to entries of the Hamiltonian $H_\alpha(\boldsymbol{x})$. For the function $z = z(\boldsymbol{x})$, we have

$$\boldsymbol{\xi}_j \cdot \mathrm{Hess}\, z(\boldsymbol{x}_0)\boldsymbol{\xi}_k = \delta_{jk}. \tag{310}$$

The second-order term in the expansion of z in terms of q_k is now written as

$$\frac{1}{2}\sum_{jk} q_j q_k \boldsymbol{\xi}_j \cdot (\mathrm{Hess}\, z(\boldsymbol{x}_0))\boldsymbol{\xi}_k = \frac{1}{2}(q_1^2 + q_2^2). \tag{311}$$

For the function $(x - iy)^2$, the second derivative with respect to s along a geodesic $\boldsymbol{x}(s)$ is evaluated at $s = 0$ or at $\boldsymbol{x}_0$ as

$$\frac{d^2}{ds^2}(x - iy)^2\bigg|_{s=0} = 2\Big(\frac{dx}{ds}(0) - i\frac{dy}{ds}(0)\Big)^2, \tag{312}$$

from which the Hessian of $(x - iy)^2$ at $\boldsymbol{x}_0$ is found to be

$$\mathrm{Hess}\,(x - iy)^2(\boldsymbol{x}_0) = \begin{pmatrix} 2 & 2i \\ 2i & -2 \end{pmatrix}. \tag{313}$$

Hence, the second order term in the expansion of $(x - iy)^2$ is written as

$$\frac{1}{2}\sum_{jk} q_j q_k \boldsymbol{\xi}_j \cdot (\mathrm{Hess}\,(x - iy)^2(\boldsymbol{x}_0))\boldsymbol{\xi}_k = (q_1 + iq_2)^2. \tag{314}$$

It turns out that the local Hamiltonian up to the second order terms in q_1, q_2 at the degeneracy point $\boldsymbol{x}_0$ for $\alpha = \frac{1}{2}$ is defined and given by

$$\begin{aligned} H_{\mathrm{loc}}(t, q; \boldsymbol{x}_0) =& t\dot{H}_{\frac{1}{2}}(\boldsymbol{x}_0) + \frac{1}{2}\sum q_j q_k \boldsymbol{\xi}_j \cdot (\mathrm{Hess} H_{\frac{1}{2}}(\boldsymbol{x}_0))\boldsymbol{\xi}_k \\ =& \begin{pmatrix} -2t + \frac{1}{4}(q_1^2 + q_2^2) & \frac{1}{2}(q_1 + iq_2)^2 \\ \frac{1}{2}(q_1 - iq_2)^2 & 2t - \frac{1}{4}(q_1^2 + q_2^2) \end{pmatrix}. \end{aligned} \tag{315}$$

We here note that the $H_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ is invariant under the action of the isotropy subgroup $SO(2)$ on the tangent plane Π_0,

$$D(h(\theta))H_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)D(h(\theta))^{-1} = H_{\mathrm{loc}}(t, h^{(2)}(\theta)q; \boldsymbol{x}_0), \tag{316}$$

where $D(h(\theta))$ and $h^{(2)}(\theta)$ are given in (302) and (308), respectively.

The eigenvalues of $H_{\mathrm{loc}}(t, q; \boldsymbol{x}_0)$ are expressed as

$$\lambda^{\pm} = \pm\sqrt{\Big(2t - \frac{1}{4}(q_1^2 + q_2^2)\Big)^2 + \frac{1}{4}\big(q_1^2 + q_2^2\big)^2}. \tag{317}$$

The degeneracy point of the present local Hamiltonian is then determined by

$$2t - \frac{1}{4}(q_1^2 + q_2^2) = 0, \quad q_1^2 + q_2^2 = 0, \tag{318}$$

to which the solution is $t = 0, q_1 = q_2 = 0$, as is expected. The exceptional point for the eigenvector associated with the positive eigenvalue λ^+ is determined by $q_1 + iq_2 = 0$, and then given by $q_1 = q_2 = 0$. According to the procedure explained in Sec. 4, the exceptional point $q = 0$ is assigned to the "up" or "down" eigenvector, according as $-2t + \frac{1}{4}(q_1^2 + q_2^2) = -2t$ is positive or negative. It then turns out that for $t < 0$ the exceptional point $q = 0$ is assigned to the "up" eigenvector and for $t > 0$ to the "down" eigenvector.

We choose to use the "up" eigenvector to evaluate the Chern number contribution from the exceptional point. Since a small circle centered at the origin $q = 0$ is oriented clockwise and since the transition function is proportional to $(q_1+iq_2)^2$, the winding number assigned to the origin for $t < 0$ is -2. We now denote the winding number for $t < 0$ and $t > 0$ by and $W_{(t<0)}$ and $W_{(t>0)}$, respectively. For $t > 0$, there is no exceptional point on Π_0 (or in the vicinity of $\boldsymbol{x}_0$ in S^2), we obtain $\Delta W = W_{(t>0)} - W_{(t<0)} = 0 - (-2) = +2$. Hence, the delta-Chern is -2; $\Delta^+ c(\boldsymbol{x}_0) = -2$. If we use the "down" eigenvector, the orientation of the small circle is anti-clockwise, so that the winding number for $t > 0$ is $+2$. Hence, we have $\Delta W = W_{(t>0)} - W_{(t<0)} = 2 - 0 = +2$ and $\Delta^+ c(\boldsymbol{x}_0) = -2$ as well. Since the delta-Chern for the eigen-line bundle associated with the negative eigenvalue λ^- is related to that for the positive eigenvalue by $\Delta^- c(\boldsymbol{x}_0) = -\Delta^+ c(\boldsymbol{x}_0)$, we have

$$\begin{pmatrix} \Delta^+ c(\boldsymbol{x}_0) \\ \Delta^- c(\boldsymbol{x}_0) \end{pmatrix} = \begin{pmatrix} -2 \\ +2 \end{pmatrix}. \tag{319}$$

Since the degeneracy point is $\boldsymbol{x}_0$ only and since the Chern number of each eigen-line bundle is zero for $0 < \alpha < \frac{1}{2}$ or for $t < 0$, we conclude that the Chern numbers of the eigen-line bundles associated with the positive and negative eigenvalue are -2 and $+2$, respectively.

With respect to the "down" eigenvector, the transition of exceptional points are depicted in Fig. 14, where the double dots attached to the maximum or the minimum point of the energy surface means that the corresponding exceptional point look like a dipole in view of the field line determined by $d(q_1 + iq_2)/d\tau = (q_1 + iq_2)^2$, where $(q_1 + iq_2)^2$ comes from the transition function.

The Chern number of the initial Hamiltonian $H_\alpha(\boldsymbol{x})$ given in (300) can be evaluated in a straightforward manner. We note here that Eq. (300) with $\alpha = 1$ is a special case of the Hamiltonian treated in [4]. For the positive eigenvalue μ^+, the exceptional point of the associated eigenvector is determined by

$$\alpha(x - iy)^2 = 0, \tag{320}$$

from which we have $x = y = 0$ for $\alpha \neq 0$. Then, one has $x = y = 0$ and $z = \pm 1$. The exceptional point is assigned to the "up" or "down" eigenvector according as $1 - \alpha + \alpha z$ is positive or negative. If $z = 1$, then $1 - \alpha + \alpha z = 1 > 0$, so that the exceptional point $(x, y, z) = (0, 0, 1) =: \boldsymbol{n}_+$ is assigned to the "up" eigenvector. If $z = -1$, then

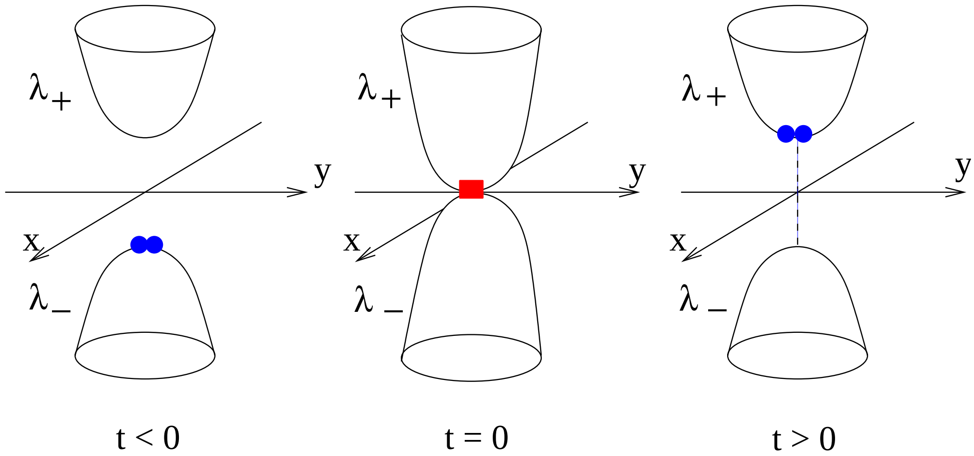

Figure 14: A schematic description of the evolution of the energy surface for (315) in a vicinity of a degeneracy point along with a variation in the control parameter passing the degeneracy value $t = 0$

$1 - \alpha + \alpha z = 1 - 2\alpha$. This implies that for $0 < \alpha < \frac{1}{2}$ only, the exceptional point $(x, y, z) = (0, 0, -1) =: \boldsymbol{n}_-$ is assigned to the "up" eigenvector, and it is not so for $\alpha > \frac{1}{2}$. The transition function is proportional to $(x - iy)^2$, which determines the winding number assigned to an exceptional point along with the orientation of a small circle centered at the exceptional point in question.

We first consider the case for $0 < \alpha < \frac{1}{2}$. Since the orientation of the coordinate system (x, y) is positive and the orientation of the small circle centered at the exceptional point $\boldsymbol{n}_+$ assigned to the "up" eigenvector is clockwise, the factor $(x - iy)^2$ gives rise to the winding number $+2$. In contrast with this, since the orientation of the coordinate system (x, y) is negative and the orientation of the small circle centered at the exceptional point $\boldsymbol{n}_-$ assigned to the "up" eigenvector is clockwise, the factor $(x - iy)^2$ gives rise to the winding number -2. Thus, the sum of the winding numbers is zero, which means that the Chern number is zero. If we choose the "down" eigenvector, we obtain the zero Chern number, since there is no exceptional point assigned.

In the case of $\frac{1}{2} < \alpha < 1$, we have the winding number $+2$ from the exceptional point $\boldsymbol{n}_+$ assigned to the "up" eigenvector. As there is no other exceptional point, the total winding number is $+2$, and hence the Chern number is -2. If we choose the "down" eigenvector, we will obtain the winding number -2 as well. Since the orientation of the coordinate system (x, y) is negative and the orientation of the small circle centered at the exceptional point $\boldsymbol{n}_-$ assigned to the "down" eigenvector is anticlockwise, the factor $(x - iy)^2$ gives rise to the winding number $+2$. Hence, the Chern number is -2.

Summing up the above discussion, we obtain the following tables;

"up"	$\alpha < \frac{1}{2}$	$\alpha > \frac{1}{2}$
$\boldsymbol{n}_+$		
$\boldsymbol{n}_-$		
Chern no.	0	-2

"down"	$\alpha < \frac{1}{2}$	$\alpha > \frac{1}{2}$
$\boldsymbol{n}_+$		
$\boldsymbol{n}_-$		
Chern no.	0	-2

(321)

Here, the symbol ═════●═════ means that the point $\boldsymbol{n}_+$ is always an exceptional point assigned to the "up" eigenvector for $0 < \alpha < 1$, and the symbol ┈┈┈┈┈○═════ that the point $\boldsymbol{n}_-$ is not an exceptional point assigned to the "down" eigenvector for $0 < \alpha < \frac{1}{2}$, but so for $\frac{1}{2} < \alpha < 1$, and further the double line indicates that the winding number contribution from the exceptional point in question is either $+2$ or -2.

If we choose the representation of $SO(2)$ as $D(h(\theta)) = \mathrm{diag}(e^{-im\theta}, e^{im\theta})$ in place of (302), we will obtain Chern numbers $\pm 2m$ for an associated Hamiltonian $H_\alpha(\boldsymbol{x})$ with $\frac{1}{2} < \alpha < 1$.

14.3 Second-order approximation at exceptional points

The Hessian formula given in Sec. 14.1 can provide an improved method for evaluating Chern number contributions from exceptional points like the exceptional points $\boldsymbol{n}_\pm$ mentioned in Sec. 5.2, at which the linearization method fails. Now we look back at Fig. 2. If we make h get close to 1, we obtain small circles $\gamma_\pm$ centered at $\boldsymbol{n}_\pm$, where $\boldsymbol{n}_\pm$ in question are exceptional points assigned to the "up" eigenvector associated with the positive eigenvalue. Then, the Chern number is given by

$$c^+ = -W(\gamma_+) - W(\gamma_-), \quad W(\gamma_\pm) = \frac{1}{2\pi i}\int_{\gamma_\pm} (\Phi^+)^{-1} d\Phi^+. \tag{322}$$

We may approximate the transition function Φ^+ in a vicinity of $\boldsymbol{n}_\pm$ in order to evaluate the winding numbers $W(\gamma_\pm)$. Since the linear approximations of Φ^+ fails at $\boldsymbol{n}_\pm$, we are recommended to take a quadratic (or higher) approximation of Φ^+. To this end, we start with the tangent planes to S^2 at $\boldsymbol{n}_\pm$. The frames $\boldsymbol{\xi}_k$ we take on the tangent planes are

$$\boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\xi}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{at} \quad \boldsymbol{n}_+ = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{323}$$

$$\boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\xi}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \quad \text{at} \quad \boldsymbol{n}_- = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}. \tag{324}$$

Applying the formulas (292) and (293) to ϕ_j given in (29), we obtain the second-order terms of the expansion of ϕ_j in terms of (q_1, q_2),

$$\psi_1(q; \boldsymbol{n}_+) := \frac{1}{2}\sum_{k,\ell} q_k q_\ell \boldsymbol{\xi}_k \cdot (\mathrm{Hess}\,\phi_1(\boldsymbol{n}_+))\boldsymbol{\xi}_\ell = -3(q_1^2 + q_2^2), \tag{325a}$$

$$\psi_2(q; \boldsymbol{n}_+) := \frac{1}{2}\sum_{k,\ell} q_k q_\ell \boldsymbol{\xi}_k \cdot (\mathrm{Hess}\,\phi_2(\boldsymbol{n}_+))\boldsymbol{\xi}_\ell = \sqrt{3}(q_1^2 - q_2^2), \tag{325b}$$

$$\psi_3(q; \boldsymbol{n}_+) := \frac{1}{2}\sum_{k,\ell} q_k q_\ell \boldsymbol{\xi}_k \cdot (\mathrm{Hess}\,\phi_3(\boldsymbol{n}_+))\boldsymbol{\xi}_\ell = q_1 q_2. \tag{325c}$$

As is easily verified, we have the same results as above for the second-order terms $\psi_j(q; \boldsymbol{n}_-)$ for ϕ_j at $\boldsymbol{n}_-$.

We proceed to the Chern number contributions from $\boldsymbol{n}_\pm$. Since the defining equation of $W(\gamma_\pm)$ is put in the form of a contour integral which takes an integer value, and since the integrand can be homotopically deformed without changing the integral values, the transition function Φ^+ can be projected to the function $a\psi_2(q;\boldsymbol{n}_+) - ib\psi_3(q;\boldsymbol{n}_+) = \sqrt{3}a(q_1^2 - q_2^2) - ibq_1q_2$ within a positive real-valued factor on the tangent plane at $\boldsymbol{n}_+$. Since the Chern number of the eigen-line bundle associated with the positive eigenvalue is viewed as a constant function on the domain $\{(a,b);\ a > 0, b > 0\}$, we may set $a = \frac{1}{\sqrt{3}}, b = 2$ for the evaluation of the Chern number contribution from $\boldsymbol{n}_+$. Then, we have $\sqrt{3}a(q_1^2 - q_2^2) - ibq_1q_2 = (q_1 - iq_2)^2$. Since the exceptional point $\boldsymbol{n}_+$ is assigned to the "up" eigenvector, the orientation of the small circle γ_+ centered at $\boldsymbol{n}_+$ is anti-clockwise, so that we see that the winding number $W(\gamma_+)$ assigned to $\boldsymbol{n}_+$ is $+2$, and hence the Chern number contribution from $\boldsymbol{n}_+$ is -2. The Chern number contribution from $\boldsymbol{n}_-$ is the same as that from $\boldsymbol{n}_+$, since the second-order terms $\psi_j(q;\boldsymbol{n}_\pm)$ have the same expression at $\boldsymbol{n}_\pm$ and since both $\boldsymbol{n}_\pm$ are assigned to the "up" eigenvector. Thus, the sum of the Chern number contributions form $\boldsymbol{n}_\pm$ are -4, *i.e.*, $c^+ = -4$ for $\{(a,b);\ a > 0, b > 0\}$. In the case of $a > 0, b < 0$, we may set $a = \frac{1}{\sqrt{3}}, b = -2$. Then, we have $\sqrt{3}a(q_1^2 - q_2^2) + ibq_1q_2 = (q_1 + iq_2)^2$. By the same reasoning as above, we obtain $c^+ = +4$. Thus we have obtained the same results on the Chern numbers as those obtained in a different method.

14.4 Delta-Chern at a triple degeneracy point

Another extension of the delta-Chern analysis is to find a way to the delta-Chern formula assigned to a degeneracy point at which eigenvalues are triply degenerate. We have already investigated such an example in Sec. 12 and inferred a possible form of the delta-Chern formula (264). However, since the model Hamiltonian (256) is of quite special type from the viewpoint of symmetry and since the triple degeneracy is much more complicate than the generic double degeneracy, the suggested formula (264) might be valid for a class of triple degeneracy points. We want to have another example supporting (264).

As a simple example of a Hamiltonian having triply degenerate eigenvalues, we take

$$H_\alpha(\boldsymbol{x}) = (1-\alpha)S_z + \alpha \boldsymbol{x}\cdot\boldsymbol{S}, \quad 1 \le \alpha \le 1, \quad \boldsymbol{x} \in S^2 \subset \mathbb{R}^3, \tag{326}$$

where S_k are the representation basis matrices for $so(3)$ with representation parameter $s = 1$. The symmetry group of this Hamiltonian is $SO(2)$, which is expressed as

$$h(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix}, \tag{327}$$

and represented as

$$D(h(\theta)) = e^{-i\theta S_3} = \begin{pmatrix} e^{-i\theta} & & \\ & 1 & \\ & & e^{i\theta} \end{pmatrix}. \tag{328}$$

The $H_\alpha(\boldsymbol{x})$ is written out as

$$H_\alpha(\boldsymbol{x}) = \begin{pmatrix} 1-\alpha+\alpha z & \frac{\alpha}{\sqrt{2}}(x-iy) & 0 \\ \frac{\alpha}{\sqrt{2}}(x+iy) & 0 & \frac{\alpha}{\sqrt{2}}(x-iy) \\ 0 & \frac{\alpha}{\sqrt{2}}(x+iy) & -1+\alpha-\alpha z \end{pmatrix} \tag{329}$$

and satisfies the symmetry condition

$$D(h(\theta))H_\alpha(\boldsymbol{x})D(h(\theta))^{-1} = H_\alpha(h(\theta)\boldsymbol{x}). \tag{330}$$

The characteristic equation for the present $H_\alpha(\boldsymbol{x})$ is given by

$$\det(\mu I - H_\alpha(\boldsymbol{x})) = \mu\big(\mu^2 - (\alpha^2(x^2+y^2) + (1-\alpha+\alpha z)^2)\big) = 0. \tag{331}$$

The degeneracy in eigenvalues occurs if and only if

$$\alpha^2(x^2+y^2) = 0, \quad 1-\alpha+\alpha z = 0. \tag{332}$$

The only solution to this equation is

$$x = y = 0, \quad z = -1, \quad \alpha = \frac{1}{2}. \tag{333}$$

This means that $\alpha = \frac{1}{2}$ is a degeneracy point in the control parameter space (or interval) $0 \le \alpha \le 1$, and $(0,0,-1)$ is the only degeneracy point on the sphere S^2. Further, the degeneracy is triple. The Chern number of the eigen-line bundles associated with the eigenvalues $1, 0, -1$ for the Hamiltonian $H_1(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{S}$ are known to be $-2, 0, 2$, respectively. This fact is proved in the context of the Berry phase (see [23], for example).

We now investigate the transition in Chern numbers by means of the linearization method applied at the degeneracy point

$$\boldsymbol{x}_0 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad \text{for} \quad \alpha = \frac{1}{2}. \tag{334}$$

The isotropy subgroup at $\boldsymbol{x}_0$ is the symmetry group $SO(2)$ itself. We choose the same frame $\boldsymbol{\xi}_k$ as (307) on the tangent plane Π_0 at $\boldsymbol{x}_0$. The Cartesian coordinates (q_k) are defined on the tangent plane through $\sum q_k \boldsymbol{\xi}_k$. The action of the isotropy subgroup on the tangent plane Π_0 is put in the same form as (308).

We first note that $H_{\frac{1}{2}}(\boldsymbol{x}_0) = 0$. The local Hamiltonian is now defined and given by

$$\begin{aligned} H_{\mathrm{loc}}(t;q;\boldsymbol{x}_0) =& t\dot{H}_{\frac{1}{2}}(\boldsymbol{x}_0) + q_1 \nabla H_{\frac{1}{2}}(\boldsymbol{x}_0)\cdot\boldsymbol{\xi}_1 + q_2 \nabla H_{\frac{1}{2}}(\boldsymbol{x}_0)\cdot\boldsymbol{\xi}_2 \\ =& \begin{pmatrix} -2t & \frac{1}{2\sqrt{2}}(q_1+iq_2) & 0 \\ \frac{1}{2\sqrt{2}}(q_1-iq_2) & 0 & \frac{1}{2\sqrt{2}}(q_1+iq_2) \\ 0 & \frac{1}{2\sqrt{2}}(q_1-iq_2) & 2t \end{pmatrix}, \end{aligned} \tag{335}$$

which is already in normal form. This local Hamiltonian satisfies the local symmetry condition

$$D(h(\theta))H_{\mathrm{loc}}(t,q;\boldsymbol{x}_0)D(h(\theta))^{-1} = H_{\mathrm{loc}}(t, h^{(2)}(\theta)q;\boldsymbol{x}_0), \tag{336}$$

where $D(h(\theta))$ and $h^{(2)}(\theta)$ are given in (328) and (308), respectively.

The characteristic equation for $H_{\rm loc}(t;q;\boldsymbol{x}_0)$ is given by

$$\det(\lambda I - H_{\rm loc}(t,q;\boldsymbol{x}_0)) = \lambda(\lambda^2 - 4t^2 - \frac{1}{4}(q_1^2+q_2^2)) = 0, \tag{337}$$

and the eigenvalues by

$$\lambda = 0, \quad \pm\sqrt{4t^2 + \frac{1}{4}(q_1^2+q_2^2)}, \tag{338}$$

Then, the degeneracy in the eigenvalues occurs if and only if

$$t = 0, \quad q_1 = q_2 = 0. \tag{339}$$

For the positive eigenvalue $\lambda^+ = \sqrt{4t^2 + \frac{1}{4}(q_1^2+q_2^2)}$, associated "up" and "down" eigenvectors are expressed as

$$\begin{pmatrix} \frac{1}{2\sqrt{2}}(q_1+iq_2)^2 \\ (\lambda^+ + 2t)(q_1+iq_2) \\ -\frac{\sqrt{2}}{4}(q_1^2+q_2^2) + 2\sqrt{2}\lambda^+(\lambda^+ + 2t) \end{pmatrix}_{\rm up}, \quad \begin{pmatrix} 2\sqrt{2}\lambda^+(\lambda^+ - 2t) - \frac{\sqrt{2}}{4}(q_1^2+q_2^2) \\ (\lambda^+ - 2t)(q_1 - iq_2) \\ \frac{1}{2\sqrt{2}}(q_1 - iq_2)^2 \end{pmatrix}_{\rm down}, \tag{340}$$

respectively. From these expressions, the exceptional points assigned to the "up" and "down" eigenvectors are shown to be determined by

$$q_1 + iq_2 = 0, \quad \lambda^+ + 2t = 0, \tag{341}$$

and by

$$q_1 - iq_2 = 0, \quad \lambda^+ - 2t = 0, \tag{342}$$

respectively. Thus, we find that according as $t < 0$ or $t > 0$ the origin $q = 0$ is an exceptional point assigned to the "up" or "down" eigenvector.

If we denote the normalized "up" and "down" eigenvectors associated with λ^+ by $|v^+(q)_{\rm up}\rangle$ and $|v^+(q)_{\rm down}\rangle$, respectively, and the respective domains of $|v^+(q)_{\rm up}\rangle$ and $|v^+(q)_{\rm down}\rangle$ by $V^+_{\rm up}$ and $V^+_{\rm down}$, then the transition function $\Phi^+(q)$ is defined through

$$|v^+(q)_{\rm up}\rangle = \Phi^+(q)|v^+(q)_{\rm down}\rangle \quad \text{on} \quad V^+_{\rm up} \cap V^+_{\rm down}. \tag{343}$$

The transition function is proportional to the ratio of the middle components of the "up" and "down" eigenvectors,

$$\frac{(\lambda^+ + 2t)(q_1+iq_2)}{(\lambda^+ - 2t)(q_1-iq_2)} = \frac{\lambda^+ + 2t}{\lambda^+ - 2t}\frac{(q_1+iq_2)^2}{q_1^2+q_2^2}. \tag{344}$$

With this function in mind, we proceed to the winding numbers assigned to exceptional points. For $t < 0$, the origin is assigned to the "up" eigenvector. According to our convention, the orientation of a small circle centered at the exceptional point assigned to the "up" eigenvector is clockwise with respect to the frame $\boldsymbol{\xi}_k$, so that the winding number, which is independent of the real factor of the transition function, is evaluated through $(q_1+iq_2)^2$ as -2, as is observed from (344). For $t > 0$, the "up" eigenvector has no exceptional points. Then, we have $W_{(t<0)} = -2$ and $W_{(t>0)} = 0$. Thus, the variation of the winding number is $\Delta W = W_{(t>0)} - W_{(t<0)} = 0 - (-2) = 2$.

The same result on ΔW is obtained by using the "down" eigenvector. The "down" eigenvector has an exceptional point at $q = 0$ for $t > 0$ only. Since the orientation of a small circle centered at the exceptional point assigned to the "down" eigenvector is anticlockwise, so that the winding number is evaluated through $(q_1 + iq_2)^2$ as $+2$. Hence, we have $\Delta W = W_{(t>0)} - W_{(t<0)} = 2 - 0 = 2$.

From the above discussion, we conclude that when the parameter t passes the degeneracy point $t = 0$ from the $t < 0$ side to the $t > 0$ side, the local delta-Chern assigned to the degeneracy point $\boldsymbol{x}_0$ is given by $\Delta^+ c(\boldsymbol{x}_0) = -2$ in association with the positive eigenvalue λ^+.

With the eigenvalue $\lambda^0 = 0$, associated are the "up" and "down" eigenvectors

$$\begin{pmatrix} \frac{1}{2\sqrt{2}}(q_1 + iq_2) \\ 2t \\ -\frac{1}{2\sqrt{2}}(q_1 - iq_2) \end{pmatrix}_{\text{up}}, \quad \begin{pmatrix} -\frac{1}{2\sqrt{2}}(q_1 + iq_2) \\ -2t \\ \frac{1}{2\sqrt{2}}(q_1 - iq_2) \end{pmatrix}_{\text{down}}, \tag{345}$$

respectively. This means that the exceptional point assigned to the "up" and the "down" eigenvectors are both the origin $q = 0$ for $t = 0$ only. Put another way, there is no exceptional point assigned to either "up" or the "down" eigenvector for $t \neq 0$. Hence, we have $\Delta W = W_{(t>0)} - W_{(t<0)} = 0$. The delta-Chern is then $\Delta^0 c(\boldsymbol{x}_0) = 0$ for the middle eigenvalue among three.

Since the negative eigenvalue λ^- is related to the positive eigenvalue λ^+ by $\lambda^- = -\lambda^+$, the similar reasoning to that for λ^+ provides the delta-Chern as $\Delta^- c(\boldsymbol{x}_0) = +2$.

The global delta-Chern is now easy to obtain. Since the orbit of the degeneracy point consists only one point, the global delta-Chen is equal to the local delta-Chern, so that we have the column of the global delta-Chern

$$\begin{pmatrix} \Delta^+ c(\boldsymbol{x}_0) \\ \Delta^0 c(\boldsymbol{x}_0) \\ \Delta^- c(\boldsymbol{x}_0) \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ +2 \end{pmatrix}, \tag{346}$$

as is expected. Since $|\Delta c| = \#\mathcal{O}_{\boldsymbol{x}_0} = 1$, it is plausible to understand that this model supports the delta-Chern formula (264) for a class of triple degeneracy points, while the symmetry group is not a finite but a continuous subgroup of $SO(3)$.

With respect to the "down" eigenvector, the transition in exceptional points is depicted in Fig. 15, where the double dots attached to the maximum or the minimum point of the energy surface means that the corresponding exceptional point look like a dipole, as in Fig. 14.

It is to be noted that this figure gives a local description of the eigenvalue evolution against the parameter t. Without a global point of view, a question arises as to the exceptional point of the "down" eigenvector of the local Hamiltonian. For $t < 0$ (*i.e.*, for $0 < \alpha < \frac{1}{2}$), the origin is an exceptional point, which means that a winding number is assigned to the origin. However, the Chern number of the eigen-line bundle associated with each eigenvalue should be zero for $0 < \alpha < \frac{1}{2}$. In order that the Chern number be zero, there must be another exceptional point to which a winding number is assigned with

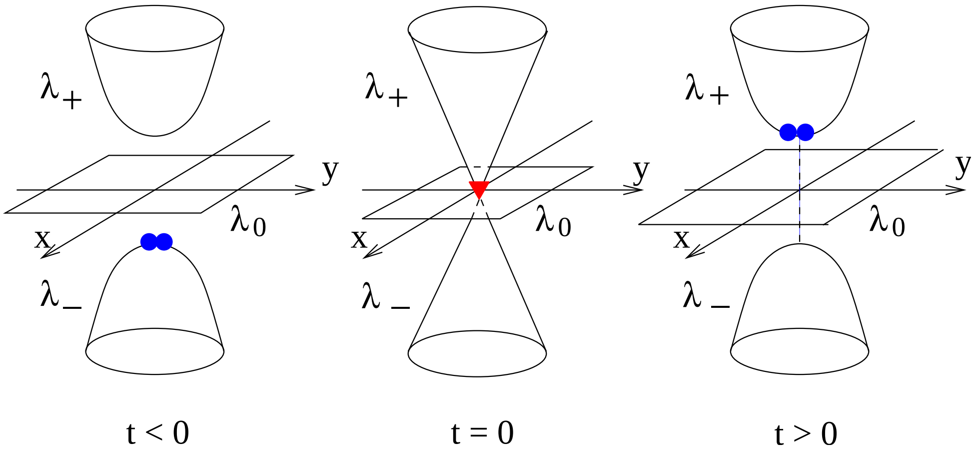

Figure 15: A schematic description of the evolution of eigenvalues in a vicinity of triple degeneracy point along with a variation in the control parameter passing the degeneracy value $t = 0$

the reverse sign. To confirm this expectation, we now return to the initial Hamiltonian. From (331), the eigenvalues of the initial Hamiltonian are given by

$$\mu = 0, \ \pm\sqrt{\alpha^2(x^2+y^2)+(1-\alpha+\alpha z)^2}. \tag{347}$$

For the positive eigenvalue $\mu^+ = \sqrt{\alpha^2(x^2+y^2)+(1-\alpha+\alpha z)^2}$, "up" and "down" eigenvectors are given by

$$\begin{pmatrix} \frac{\alpha^2}{2}(x-iy)^2 \\ \left(\mu^+ - (1-\alpha+\alpha z)\right)\frac{\alpha}{\sqrt{2}}(x-iy) \\ -\frac{\alpha^2}{2}(x^2+y^2)+\mu^+(\mu^+-(1-\alpha+\alpha z)) \end{pmatrix}_{\mathrm{up}}, \quad \begin{pmatrix} \mu^+(\mu^+ + 1-\alpha+\alpha z) - \frac{\alpha^2}{2}(x^2+y^2) \\ (\mu^+ + 1-\alpha+\alpha z)\frac{\sqrt{2}}{2}(x+iy) \\ \frac{\alpha^2}{2}(x+iy)^2 \end{pmatrix}_{\mathrm{down}}, \tag{348}$$

respectively.

Exceptional points assigned to the "up" eigenvector are then determined by

$$x - iy = 0, \quad \mu^+ - (1-\alpha+\alpha z) = 0. \tag{349}$$

From $x - iy = 0$, we obtain $x = y = 0$, and hence $z = \pm 1$. If $z = 1$, then $\mu^+ = 1$, so that $\mu^+ - (1-\alpha+\alpha z) = 0$. Thus, $(x, y, z) = (0, 0, 1) =: \boldsymbol{n}_+$ is an exceptional point. If $z = -1$, then $\mu^+ = |1-2\alpha|$, and further $\mu^+ - (1-\alpha+\alpha z) = |1-2\alpha| - (1-2\alpha)$. This implies that for $0 < \alpha < \frac{1}{2}$, $\mu^+ - (1-\alpha+\alpha z) = 0$ holds, so that $(x, y, z) = (0, 0, -1) =: \boldsymbol{n}_-$ is another exceptional point for $0 < \alpha < \frac{1}{2}$, and further for $\frac{1}{2} < \alpha < 1$, the point $\boldsymbol{n}_-$ is not an exceptional point.

Exceptional points assigned to the "down" eigenvector are determined by

$$\mu^+ + 1 - \alpha + \alpha z = 0, \quad x + iy = 0. \tag{350}$$

From $x + iy = 0$, we obtain $x = y = 0$, and hence $z = \pm 1$. If $z = 1$, then $\mu^+ = 1$, and hence $\mu^+ + 1 - \alpha + \alpha z = 2$. This means that $(x, y, z) = (0, 0, 1) =: \boldsymbol{n}_+$ is not an exceptional point for the "down" eigenvector. If $z = -1$, then $\mu^+ + 1 - \alpha + \alpha z = |1 - 2\alpha| + 1 - 2\alpha$. This implies that $(x, y, z) = (0, 0, -1) = \boldsymbol{n}_-$ is an exceptional point assigned to the "down" eigenvector for $\frac{1}{2} < \alpha < 1$ only.

The transition function is proportional to the ratio of middle elements of the "up" and "down" eigenvectors,

$$\frac{\mu^+ - (1 - \alpha + \alpha z)}{\mu^+ + 1 - \alpha + \alpha z} \frac{x - iy}{x + iy} = \frac{\mu^+ - (1 - \alpha + \alpha z)}{\mu^+ + 1 - \alpha + \alpha z} \frac{(x - iy)^2}{x^2 + y^2}. \tag{351}$$

Since the real factor of the transition function is irrelevant to the winding number, the winding number is determined by the factor $(x - iy)^2$ together with the orientation of the coordinate system (x, y) at the exceptional point in question and the orientation of a small circle centered at the exceptional point.

Although the Hamiltonian treated in this subsection is different from that treated in Sec. 14.2, the reasoning to be made in the following is quite the same as those done in the latter part of Sec. 14.2 except for the last paragraph. We then obtain the same result as (321) on the Chern number of the eigen-line bundle associated with the positive eigenvalue μ^+, which shows that the Chern number for $0 < \alpha < \frac{1}{2}$ is zero.

15 Concluding remarks

In the previous papers [4, 5], we have observed from the two-level model Hamiltonians with symmetry that the possible values of Chern numbers of the eigen-line bundles associated with respective eigenvalues are closely related to representations of the symmetry group. From the same point of view, we now have another look at the results on Chern numbers realized on our model Hamiltonians.

Let (J) denote the representation of $SO(3)$ labeled by J, which is considered as a reducible representation of the finite O group. Then, the tensor product of the (J) representation of $SO(3)$ and the F_i, $i = 1, 2$, representation of the O group is decomposed into the direct sum of the form

$$F_i \otimes (J) = (J + \Delta_{\text{max}})_i \oplus (J + \Delta_{\text{middle}})_i \oplus (J + \Delta_{\text{min}})_i, \tag{352}$$

where

$$\Delta_{\text{max}} + \Delta_{\text{middle}} + \Delta_{\text{min}} = 0, \tag{353}$$

and where $(J + \Delta_k)_i$, $k = \text{max}, \text{middle}, \text{min}$, denote for short $(J + \Delta_k) \otimes A_i, i = 1, 2$, with A_i referring to the representation of the O group. Possible pairs of values $(\Delta_{\text{max}} - \Delta_{\text{min}}, \Delta_{\text{middle}})$ are given in Fig. 16. Some of possible values of $\Delta_k, k = \text{max}, \text{midle}, \text{min}$, are given in Tables 10 and 11 in [3], pp. 117-118, for low values of J.

From Fig. 16, we pick up several sets of values of Δ_k with $\Delta_{\text{middle}} \neq 0$,

$$\begin{pmatrix} \Delta_{\text{max}} \\ \Delta_{\text{middle}} \\ \Delta_{\text{min}} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \\ -7 \end{pmatrix}, \begin{pmatrix} 7 \\ -1 \\ -6 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix}. \tag{354}$$

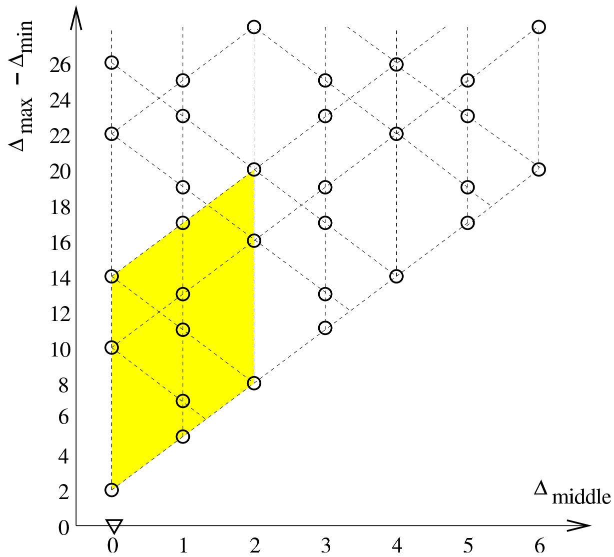

Figure 16: Small circles stand for possible pairs of $(\Delta_{\max}-\Delta_{\min}, \Delta_{\text{middle}})$ corresponding to the decomposition (352) subject to (353). As the figure is symmetric with respect to the reflection $\Delta_{\text{middle}} \to -\Delta_{\text{middle}}$, only positive values of Δ_{middle} are shown. Broken lines are drawn to make the figure readable. A characteristic pattern of the lattice is that the lattice shown above is periodic with an elementary cell with vertices $(0,2),(0,14),(2,20),(2,8)$.

If doubled, each of the above columns of possible values of Δ_i is in one-to-one correspondence to one of columns of the Chern numbers shown in Fig. 7 and Fig. 8 as sets of numbers. If $\Delta_{\text{middle}}=0$, possible values of $\Delta_{\max}-\Delta_{\min}$ are shown to be $12j\pm 2$, $j=0,1,2,\ldots,$ as is seen from Fig. 16. A set of possible values of Δ_k with $\Delta_{\text{middle}}=0$

$$\begin{pmatrix}\Delta_{\max}\\ \Delta_{\text{middle}}\\ \Delta_{\min}\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}5\\0\\-5\end{pmatrix}, \begin{pmatrix}7\\0\\-7\end{pmatrix} \tag{355}$$

are, if doubled, realized as Chern numbers shown in Fig. 13. Though the model Hamiltonian (256) is of special type from the viewpoint of symmetry, it is worth pointing out that the Chern numbers shown in Fig. 13 can be treated on the same footing as that for Chern numbers associated with the Hamiltonian (19) of generic type.

These correspondences seem to be rather formal, since no line bundle appears in (352). However, if we view the operators concerning with (J) representation as classical variables and as forming the sphere, the right-hand side of (352) may be considered as corresponding to the direct sum of eigen-line bundles over the sphere, and A_1 or A_2 may be looked upon as a representation of the O group acting on sections of respective eigen-line bundles.

In the same picture, we look again at the Chern numbers obtained in Sec. 14. Let $(m)_{SO(2)}$ and $(J)_{SO(3)}$ denote the irreducible representations of $SO(2)$ and $SO(3)$ labeled by m and J, respectively, where $(J)_{SO(3)}$ is viewed as a reducible representation of $SO(2)$; $(J)_{SO(3)} = \sum_{m=-J}^{J}(m)_{SO(2)}$. Then, like (352), we have the decomposition of the tensor product representation

$$\big((-m)_{SO(2)} \oplus (m)_{SO(2)}\big) \otimes (J)_{SO(3)} = (J-m)_{SO(3)} \oplus (J+m)_{SO(3)}, \tag{356}$$

where $(J-m)_{SO(3)}$ and $(J+m)_{SO(3)}$ should be understood as the tensor products with the trivial representation of $SO(2)$, $(J-m)_{SO(3)} \otimes (0)_{SO(2)}$ and $(J+m)_{SO(3)} \otimes (0)_{SO(2)}$, respectively. From this decomposition, we expect that the corresponding Chern numbers are $(-2m, 2m)$. On the model Hamiltonian (300), we have realized the Chern numbers $(-2, 2)$ with $m = 1$.

In a similar manner, we have the decomposition

$$\big((-1)_{SO(2)} \oplus (0)_{SO(2)} \oplus (1)_{SO(2)}\big) \otimes (J)_{SO(3)} = (J-1)_{SO(3)} \oplus (J)_{SO(3)} \oplus (J+1)_{SO(3)}. \tag{357}$$

If we rewrite the left-hand side as $(1)_{SO(3)} \otimes (J)_{SO(3)}$, the above decomposition is exactly the same as the Clebsch-Gordan formula for the $SO(3)$ (or $SU(2)$) representation. The expected Chern numbers $(-2, 0, 2)$ are realized on the model Hamiltonian (329).

While we have applied the delta-Chern formula to two- and there-level semi-quantum systems, the delta-Chern formula can be applied to n-level semi-quantum systems, as was remarked after Thm. 9.1. A formal application of the delta-Chern formula to a five-level model is found in [6], where the model comes from the tetrahedral molecule SiH_4.

We have observed that the possible values of Chern numbers of the eigen-line bundles associated with respective eigenvalues are closely related to representations of the symmetry group for the three-level model Hamiltonians with symmetry in the present article as well as for the two-level model Hamiltonians in the preceding papers [4, 5]. Though a number of supporting results have been accumulated, the systematic proof of the marked correspondence between possible Chern numbers and possible decomposition of the tensor product representation remains to be open. Another big step for extending the present Chern number analysis consists in a generalization to fiber bundles defined over higher dimensional base space such as $\mathbb{C}P^n$. Example of such a three-level model over $\mathbb{C}P^2$ was studied in [24].

After finishing the present work, we have achieved a progress in the study of band rearrangements in comparison between full quantum and semi-quantum systems. As is pointed out in Introduction, the delta-Chern can be interpreted as describing a band rearrangement. This statement has been confirmed by introducing a Dirac operator corresponding to a linearized Hamiltonian at a degeneracy point of S^2. A local delta-Chern for the linearized semi-quantum Hamiltonian and an extended notion of spectral flow for the full quantum Dirac operator are in fine correspondence [25, 26].

A Cubic symmetry

Let $\boldsymbol{e}_k, k = 1, 2, 3$, denote the standard basis of $\mathbb{R}^3$. We take six points $\pm\boldsymbol{e}_k$ as vertices of the regular octahedron. Each vertex is given a number as follows: 1,2,3 are assigned to $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, respectively, and 4,5,6 to $-\boldsymbol{e}_1, -\boldsymbol{e}_2, -\boldsymbol{e}_3$, respectively. The orientation-preserving symmetry group for the regular octahedron is called the octahedral rotation group and is denoted by O. The O group is known to be isomorphic with the symmetric group S_4 and the order of O is 24. Each of the elements has the symbol to denote the assigned rotation, which, except for the identity, are

$$\begin{gathered}\{C_4^X, C_4^{-X}, C_4^Y, C_4^{-Y}, C_4^Z, C_4^{-Z}\},\\ \{C_3^{[111]}, C_3^{[-11-1]}, C_3^{[1-1-1]}, C_3^{[-1-11]}, C_3^{[-1-1-1]}, C_3^{[1-11]}, C_3^{[-111]}, C_3^{[11-1]}\},\\ \{C_2^X, C_2^Y, C_2^Z\},\\ \{C_2^{[011]}, C_2^{[01-1]}, C_2^{[101]}, C_2^{[-101]}, C_2^{[110]}, C_2^{[-110]}\},\end{gathered} \tag{358}$$

where the elements in each parentheses are conjugate to one another, and where C_4^X denote the counterclockwise rotation about the X-axis by the angle $2\pi/4$, $C_3^{[111]}$ the counterclockwise rotation about the $(1,1,1)^T$-axis by $2\pi/3$, C_2^X the counterclockwise rotation about X-axis by $2\pi/2$, and $C_2^{[011]}$ the counterclockwise rotation about the $(0,1,1)^T$-axis by $2\pi/2$. In Fig. 17, the axes for $C_4^Z, C^{[111]}$, and $C_2^{[110]}$ are described. The other symmetry axes, which are not drawn, can be easily found.

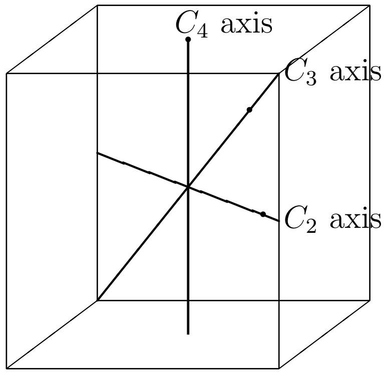

Figure 17: Representative symmetry axes. The C_4, C_3, and C_2 axes shown in the figure correspond to C_4^Z, $C_3^{[111]}$, and $C_2^{[110]}$, respectively. Since the centers of the respective faces of the regular cube are the vertices of the regular octahedron, the orientation-preserving symmetry group of the regular cube is isomorphic with the octahedral group.

We have already the matrix expressions of the C_4^Z and $C_3^{[-1-1-1]}$ in (12), which we denote by ζ and τ, respectively, in this Appendix,

$$\zeta = \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}. \tag{359}$$

As is mentioned in Sec. 2, the O group is generated by ζ and τ. For example, $C_2^{[110]}$ has the matrix expression which is generated by ζ and τ as

$$C_2^{[110]} \mapsto \begin{pmatrix} & 1 & \\ 1 & & \\ & & -1 \end{pmatrix} = \tau\zeta\tau. \tag{360}$$

We now describe a few of the representation of the O group. The O group generated by ζ and τ acts on $\mathbb{R}^3$ and naturally induces its representation on the linear span of the monomials x, y, and z, which is called a T_1 representation. In a similar manner, the linear span of the monomials

$$X = yz, \quad Y = zx, \quad Z = xy \tag{361}$$

form a three dimensional (T_2-representation) space for the O group. Under the actions of ζ and by τ on $\mathbb{R}^3$, these basis polynomials transform according to

$$(X, Y, Z) \mapsto (Y, -X, -Z), \quad (X, Y, Z) \mapsto (Y, Z, X), \tag{362}$$

respectively. For another example, let

$$\phi_1 = 2z^2 - x^2 - y^2, \quad \phi_2 = \sqrt{3}(x^2 - y^2). \tag{363}$$

The linear span of ϕ_1 and ϕ_2 forms a two-dimensional representation (or E-representation) space for the O group. Under the actions by ζ and by τ on $\mathbb{R}^3$, the basis polynomials transform according to

$$\phi_1 \mapsto \phi_1, \quad \phi_2 \mapsto -\phi_2, \tag{364a}$$

$$\phi_1 \mapsto -\frac{1}{2}\phi_1 + \frac{\sqrt{3}}{2}\phi_2, \quad \phi_2 \mapsto -\frac{\sqrt{3}}{2}\phi_1 - \frac{1}{2}\phi_2, \tag{364b}$$

respectively. This gives rise to the representation matrices given in (13).

The group O acts on $\mathbb{R}^3$ as matrices and also on 3×3 matrices by adjoint action. Since the O is a subgroup of $SO(3)$, the O is considered as acting on the sphere S^2. For the Hamiltonian (19), the symmetry condition (1) takes the form

$$gH(\boldsymbol{x})g^{-1} = H(g\boldsymbol{x}) \quad \text{for} \quad g \in G, \tag{365}$$

where G denote the group O whose elements g's are matrices generated by ζ and τ given in (359) or in (12). In order to verify (365), we put the Hamiltonian (19) in the form of a linear combination of the three Hermitian matrices $H_{\mathrm{a}}, H_{\mathrm{d}}, H_{\mathrm{s}}$, where

$$H_{\mathrm{a}}(\boldsymbol{x}) = \begin{pmatrix} 0 & -iz & iy \\ iz & 0 & -ix \\ -iy & ix & 0 \end{pmatrix}, \tag{366a}$$

$$H_{\mathrm{d}}(\boldsymbol{x}) = \begin{pmatrix} 2x^2 - y^2 - z^2 & & \\ & 2y^2 - z^2 - x^2 & \\ & & 2z^2 - x^2 - y^2 \end{pmatrix}, \tag{366b}$$

$$H_{\mathrm{s}}(\boldsymbol{x}) = \begin{pmatrix} 0 & xy & zx \\ xy & 0 & yz \\ zx & yz & 0 \end{pmatrix}. \tag{366c}$$

Since the octahedral group is generated by ζ and τ, and since the Hamiltonian is the linear combination of $H_{\mathrm{a}}, H_{\mathrm{d}}, H_{\mathrm{s}}$, it is sufficient for us to check whether the symmetry condition (1) holds or not, for $H_{\mathrm{a}}, H_{\mathrm{d}}, H_{\mathrm{s}}$ and ζ, τ only (see [6] for tensorial expressions of these Hamiltonians). These procedures can be easily carried out.

For the Hamiltonian (18), the symmetry condition (1) takes the form

$$D(g)H(\boldsymbol{x})D(g)^{-1} = H(g\boldsymbol{x}), \tag{367}$$

where $D(g)$ are matrices (see (13)) generated by

$$D(\zeta) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad D(\tau) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}. \tag{368}$$

The procedure to show the invariance (367) is similar to that for (365)

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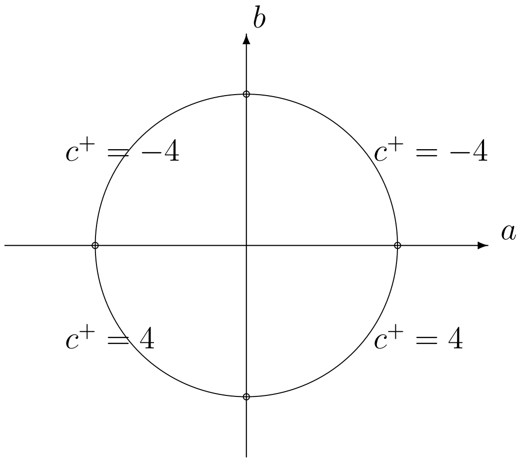

Figure 1: Chern numbers assigned to arcs of the unit circle

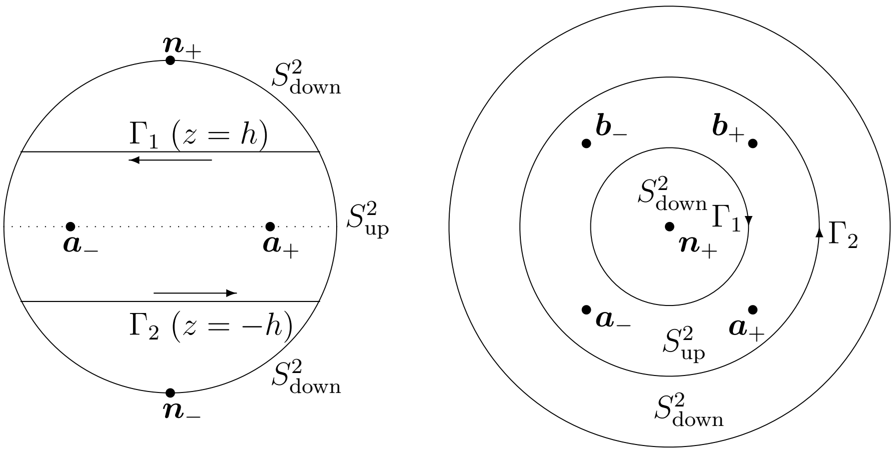

Figure 2: division of the sphere into the disjoint union of $S^2_{\rm up}$ and $S^2_{\rm down}$ with $\omega^+_{\rm up}$ and $\omega^+_{\rm down}$ being smoothly defined on $S^2_{\rm up}$ and $S^2_{\rm down}$, respectively

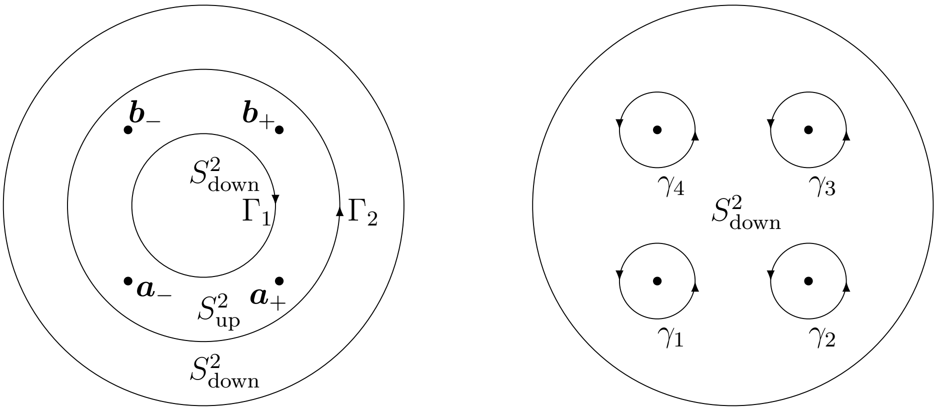

Figure 3: Deformation of contours $\Gamma_i, i = 1, 2$, into small circles $\gamma_j, j = 1, 2, 3, 4$, centered at exceptional points

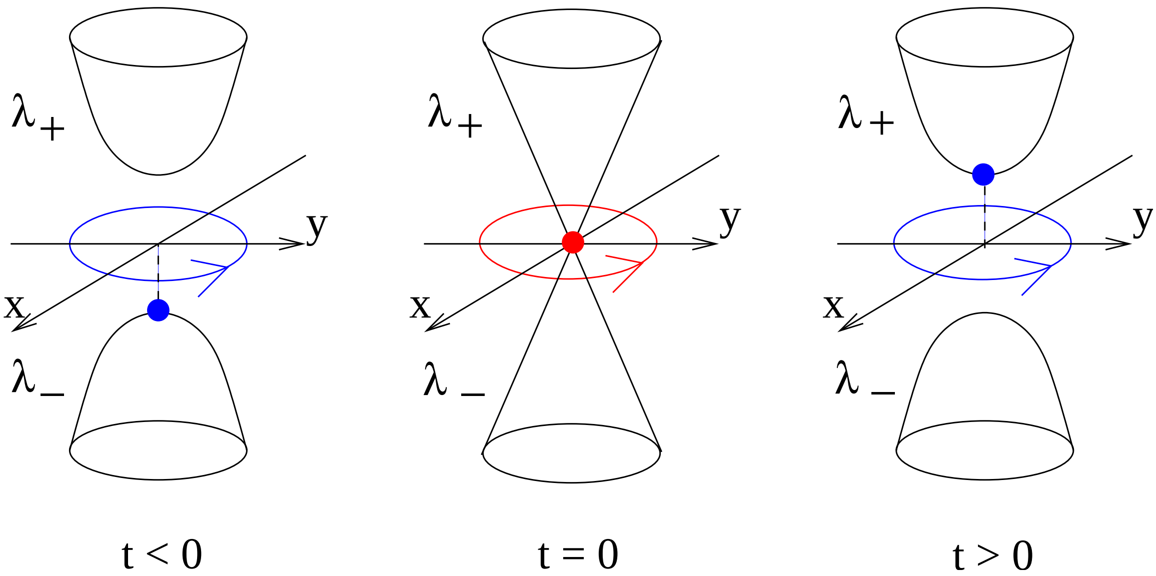

Figure 4: A schematic representation of the evolution of eigenvalues of a local linearized model Hamiltonian in a two-level approximation along with variation of a control parameter t crossing the boundary of the iso-Chern domain. Exceptional points (blue points) assigned to the "down" eigenvector are shown in the λ_+ $(t > 0)$ and λ_- $(t < 0)$ components.

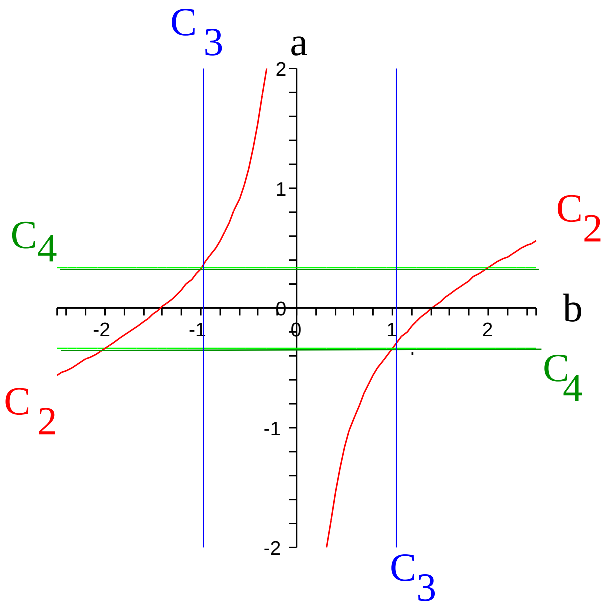

Figure 5: Degeneracy curves in the space of control parameters for the Hamiltonian (19).

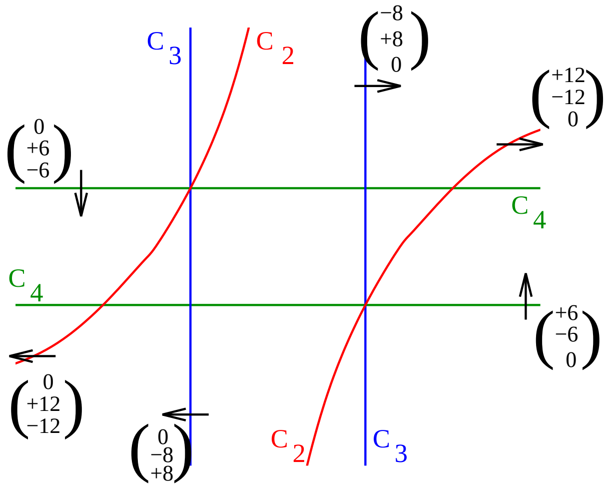

Figure 6: Delta-Chern diagram for the Hamiltonian (19).

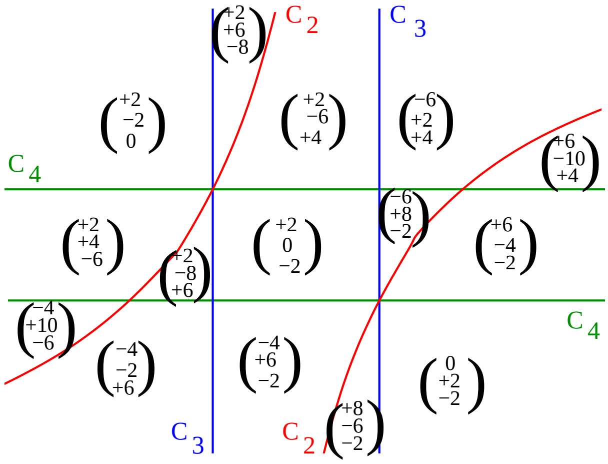

Figure 7: Iso-Chern diagram for the Hamiltonian (19).

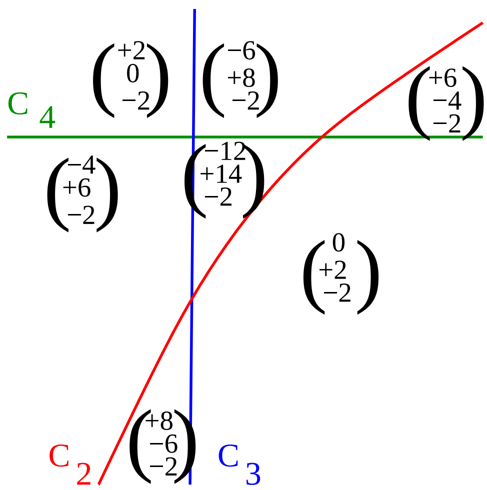

Figure 8: A part of the Chern diagram for the Hamiltonian (213) with $c = \frac{1}{2}$ is zoomed in.

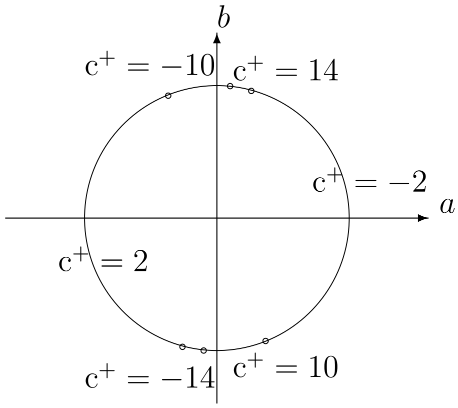

Figure 9: Chern numbers c^+ of the eigen-line bundle associated with the positive eigenvalue are assigned to arcs separated by degeneracy points.

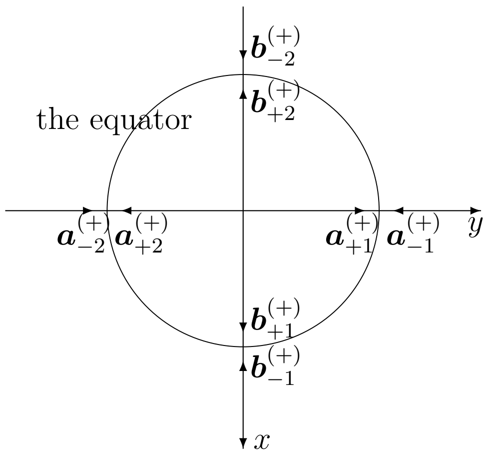

Figure 10: As $\varepsilon \to -\frac{5}{2}$, the exceptional points $\boldsymbol{a}_{\pm j}^{(+)}$ and $\boldsymbol{b}_{\pm j}^{(+)}$ are getting together to be degeneracy points and then to vanish pairwise for $\varepsilon > -\frac{5}{2}$.

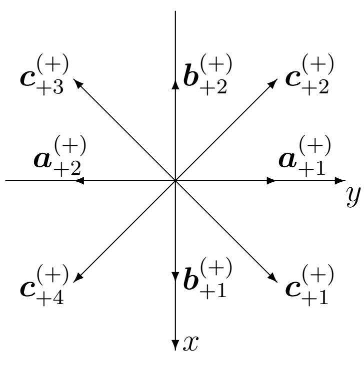

Figure 11: When the parameter value passes $\varepsilon = \frac{5}{3}$ upward, new exceptional points $\boldsymbol{a}_{+j}^{(+)}$, $\boldsymbol{b}_{+j}^{(+)}$, and $\boldsymbol{c}_{+k}^{(+)}$ branch off from the north pole $\boldsymbol{n}_{+}^{(+)}$.

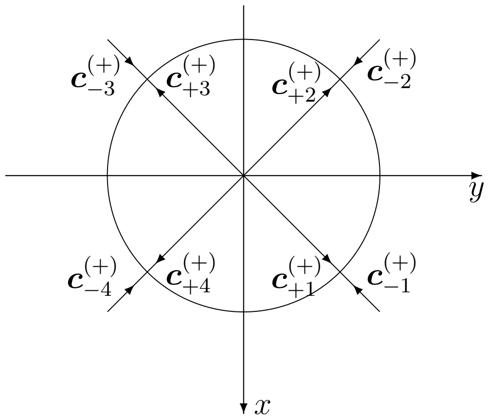

Figure 12: As $\varepsilon \to 10$, the exceptional points $\boldsymbol{c}^{(+)}_{\pm k}$ are getting together pairwise, and vanish for $\varepsilon > 10$.

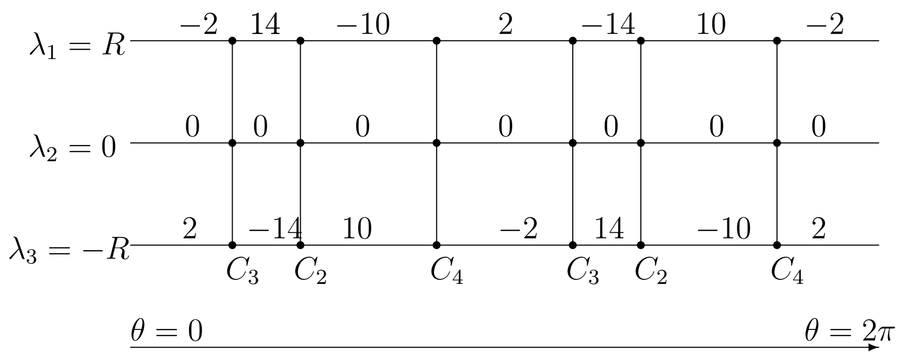

Figure 13: Chern number variation for the Hamiltonian (256)

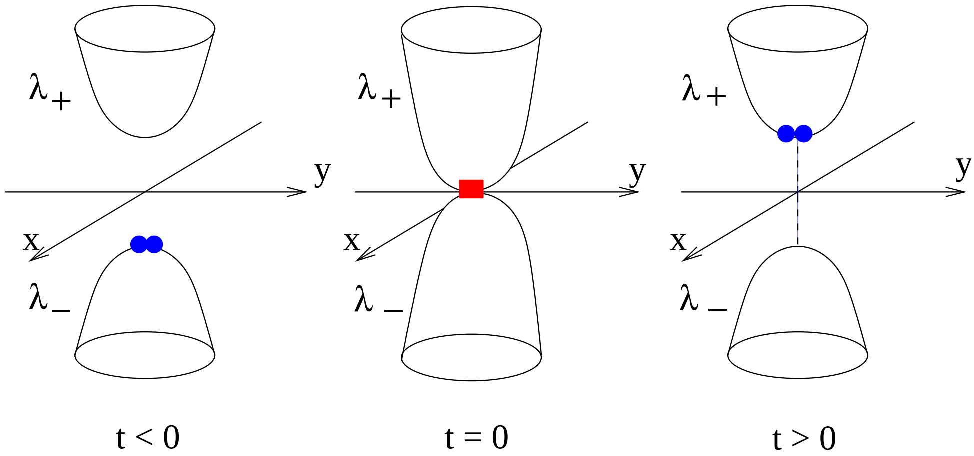

Figure 14: A schematic description of the evolution of the energy surface for (315) in a vicinity of a degeneracy point along with a variation in the control parameter passing the degeneracy value $t = 0$

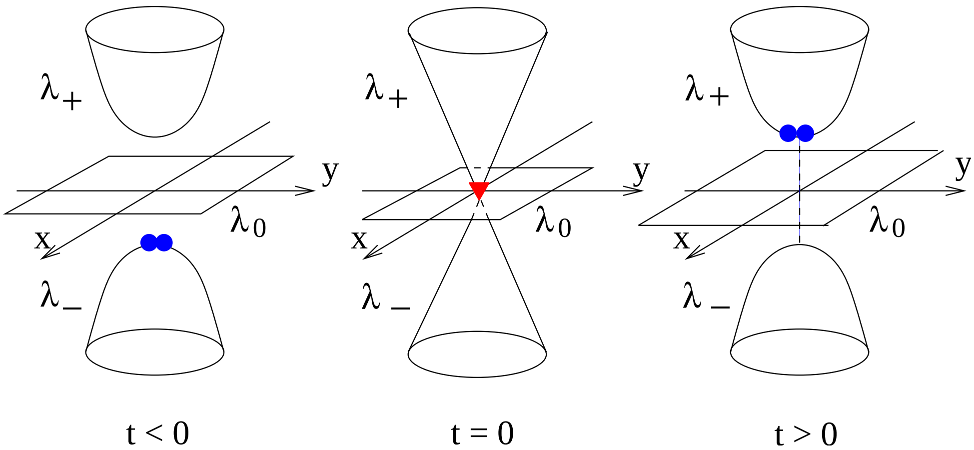

Figure 15: A schematic description of the evolution of eigenvalues in a vicinity of triple degeneracy point along with a variation in the control parameter passing the degeneracy value $t = 0$

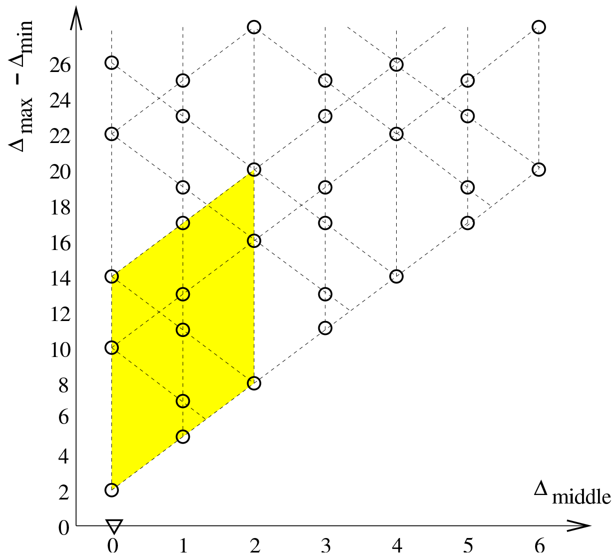

Figure 16: Small circles stand for possible pairs of $(\Delta_{\rm max} - \Delta_{\rm min}, \Delta_{\rm middle})$ corresponding to the decomposition (352) subject to (353). As the figure is symmetric with respect to the reflection $\Delta_{\rm middle} \to -\Delta_{\rm middle}$, only positive values of $\Delta_{\rm middle}$ are shown. Broken lines are drawn to make the figure readable. A characteristic pattern of the lattice is that the lattice shown above is periodic with an elementary cell with vertices $(0, 2), (0, 14), (2, 20), (2, 8)$.

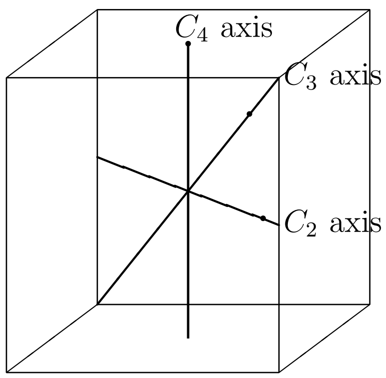

Figure 17: Representative symmetry axes. The C_4, C_3, and C_2 axes shown in the figure correspond to C_4^Z, $C_3^{[111]}$, and $C_2^{[110]}$, respectively. Since the centers of the respective faces of the regular cube are the vertices of the regular octahedron, the orientation-preserving symmetry group of the regular cube is isomorphic with the octahedral group.