Qualitative tools in natural science. Symmetry, topology, complexity.

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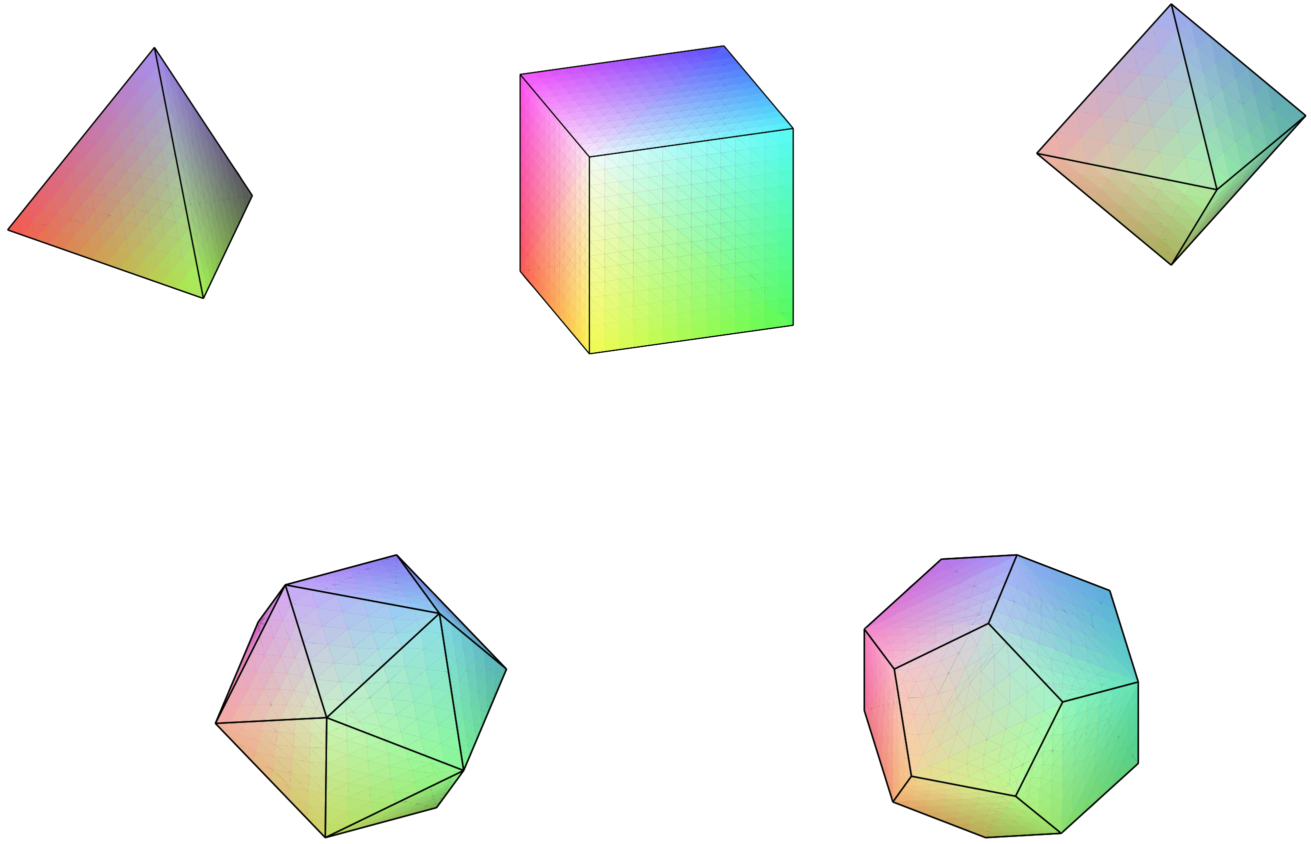
“The great book of the Universe stays open before our eyes; but to understand it, we have first to learn the language in which it is written, the *mathematics*.”

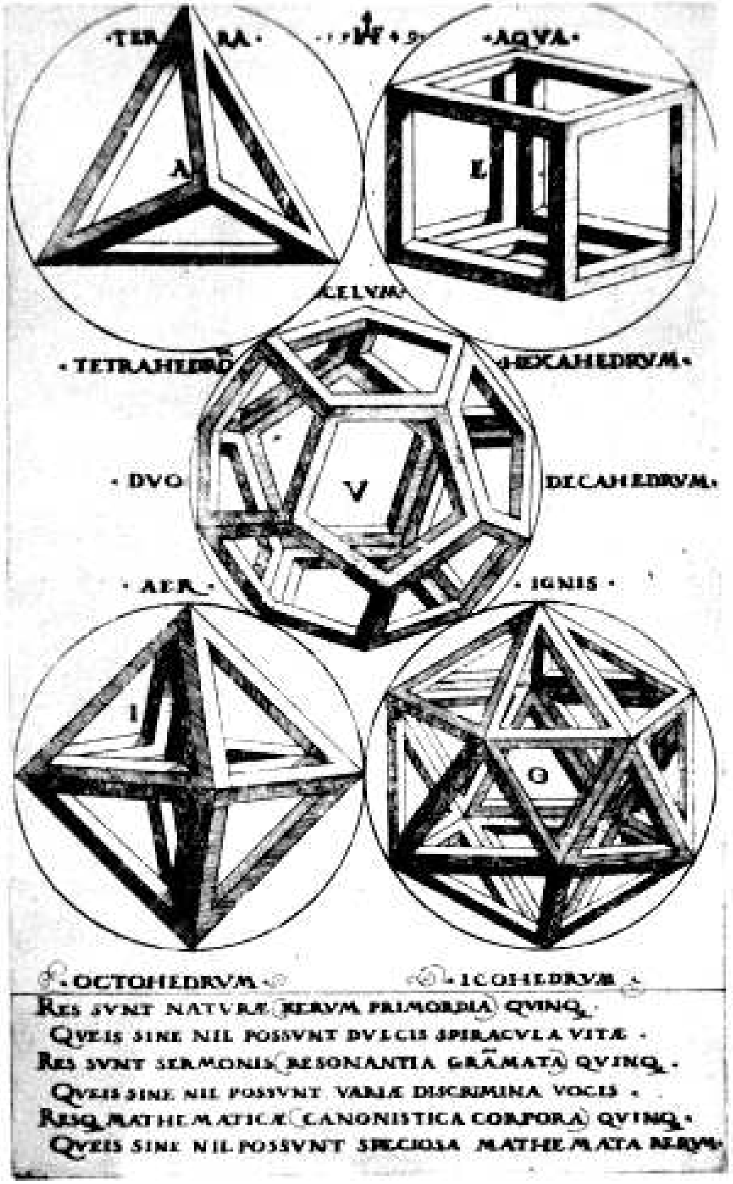
G**alileo Galilei** (1564-1642)

What mathematics to use?

J.L. Lagrange (1736 - 1813) J. J. Rousseau (1712 - 1778)

Regular Polyhedrons. (Platon bodies)





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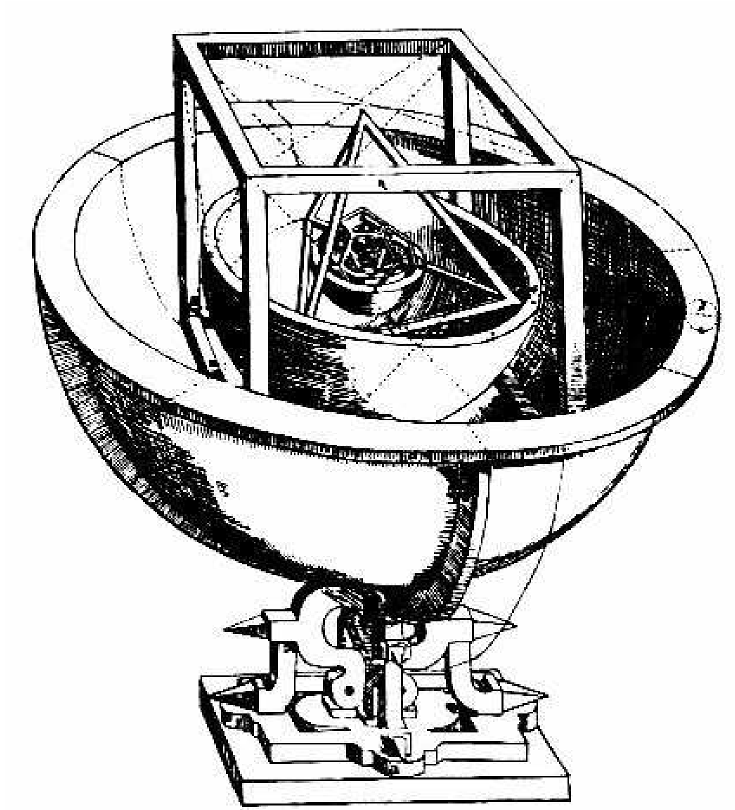
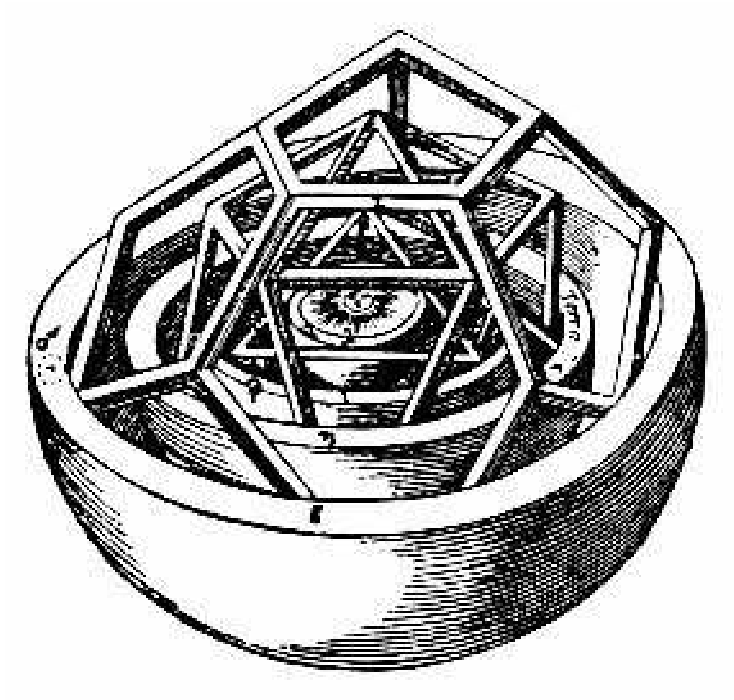
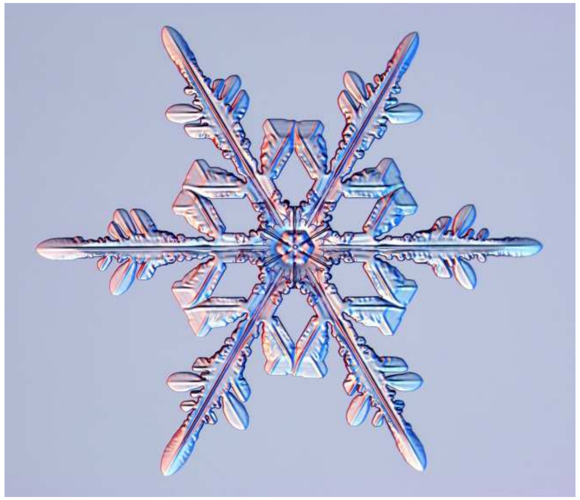
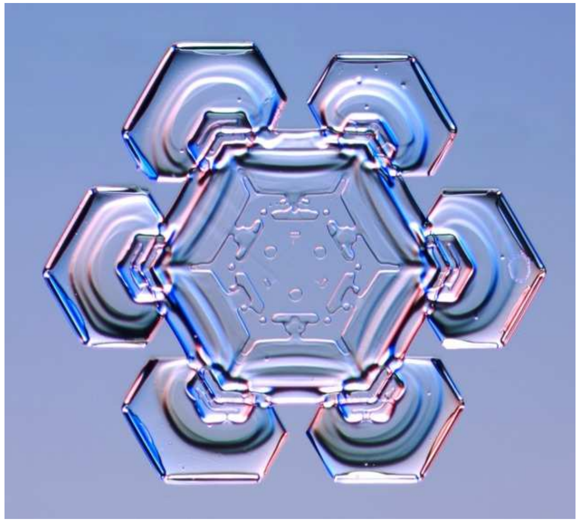
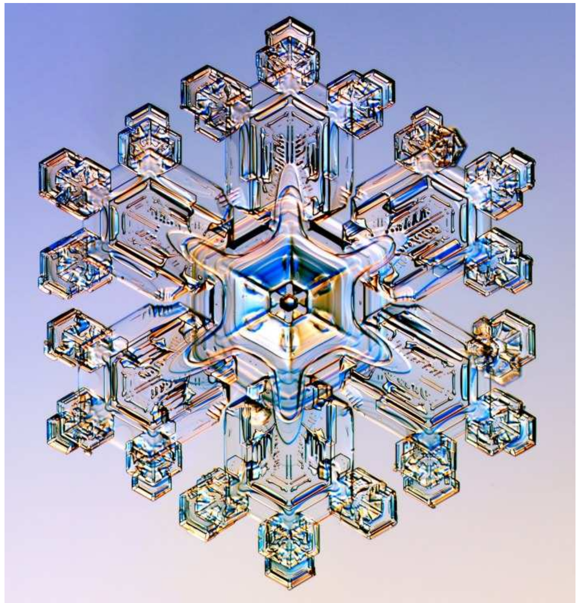
·OCTOHEDRVM·
·ICOHEDRVM·
RES SVNT NATVRÆ (RERVM PRIMORDIA) QVINQ.
QVEIS SINE NIL POSSVNT DVLCIS SPIRACVLA VITÆ ·
RES SVNT SERMONIS (RESONANTIA GRĀMATA) QVINQ ·
QVEIS SINE NIL POSSVNT VARIÆ DISCRIMINA VOCIS ·
RESQ MATHEMATICÆ (CANONISTICA CORPORA) QVINQ ·
QVEIS SINE NIL POSSVNT SPECIOSA MATHEMATA RERVM·

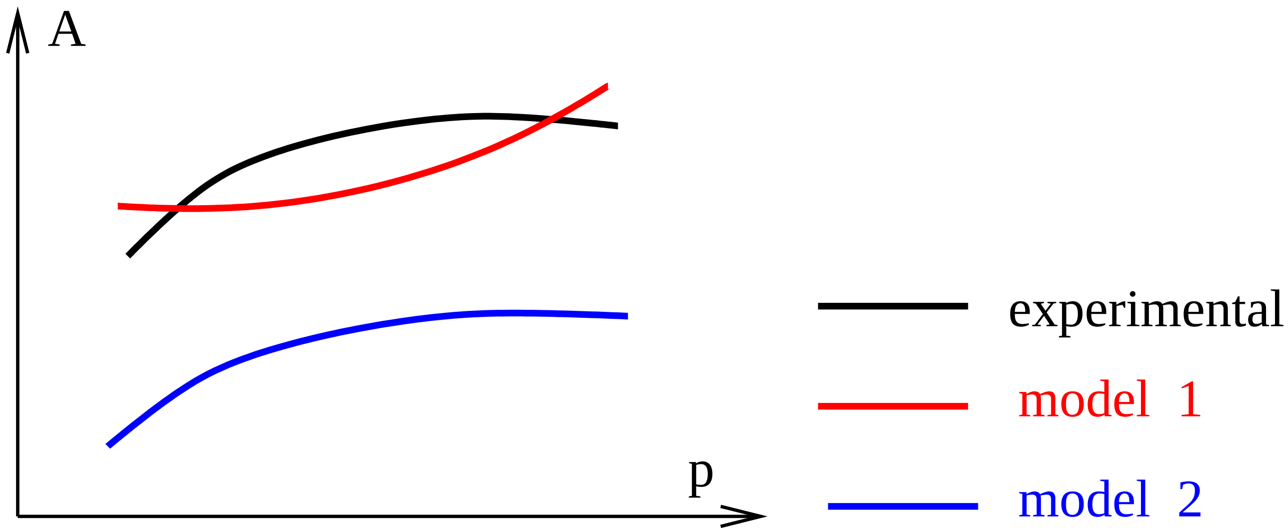
Johannes Kepler (1571 - 1630)

Renè Thom (1923 - 2002)

[Structural Stability and Morphogenesis, W.A.Benjam, 1972]

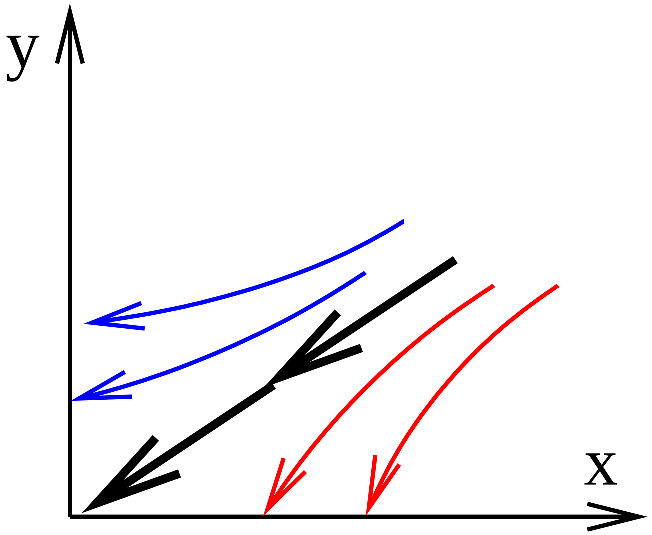


What model describes better the behavior of quantity A as a function of control parameter p ?

Good simulation should include as **much** detail as possible.

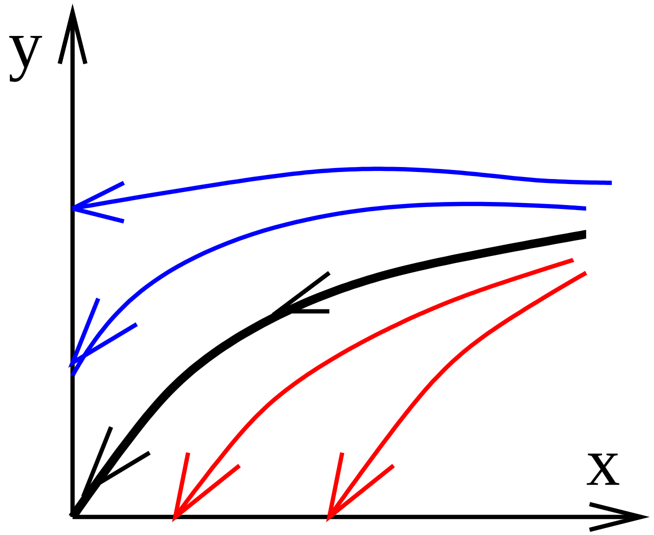
Good model should include as **little** detail as possible.

Model of war.



$$\dot{x} = -by,$$
$$\dot{y} = -ax,$$

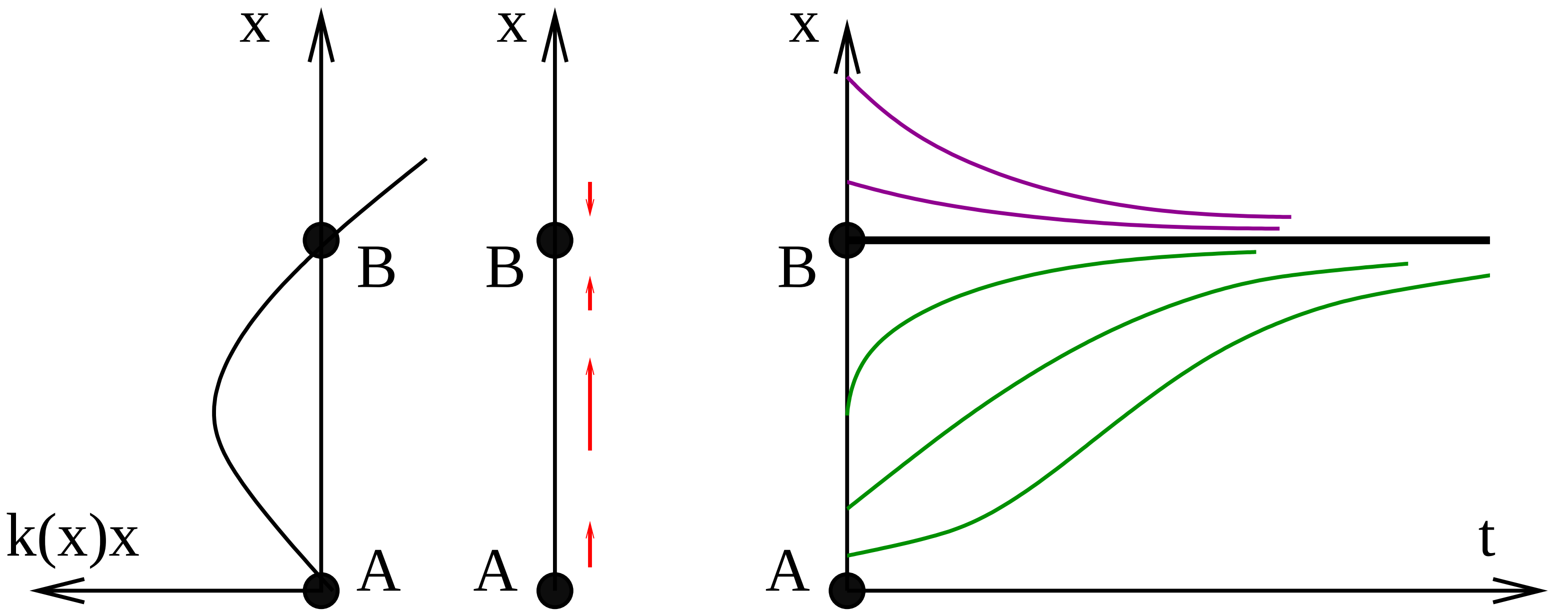
“rigid“



$$\dot{x} = -b(x, y)y,$$
$$\dot{y} = -a(x, y)x,$$

“soft“

Logistic model.



$$\dot{x} = k(x)x; \qquad k(x) = ax - bx^2$$

Table 1: Frequency of appearence as first digit of numbers in 2^n, $n \in \mathbb{N}^*$ sequence.

First digit	1	2	3	4	5	6	7	8	9
Frequency	0,301	0,176	0,125	0,097	0,079	0,067	0,058	0,051	0,046

Combinatorial tools

1. Lucky tickets.
2. Generating functions.
3. Fibonacci numbers and their occurence in Nature.
4. Gold section (“Divine proportion“)

Lucky tickets

“Lucky“	“Unlucky“
123 222	123 234
264 156	531 777

Let a_n be the number of triples giving the sum of digits equal n.

$a_0 = 1$ (only 000 has sum 0),

$a_1 = 3$ (there are three triples 001, 010, 100),

$a_2 = 6$ (triples 002, 020, 200, 011, 101, 110), etc.

If we know a_n, the sum of lucky tickets is $\sum a_n^2$. We need to find 28 numbers and then to calculate the sum of their squares.

Construction through generating functions

$$A_1(s) = 1 + s + s^2 + \cdots + s^9$$

This polynomial has the following symbolic meaning: The coefficient at s^n is equal to the number of one-digit numbers with the sum of digits equal n.

Now the polynomial A_2 should be of degree 18.

$$A_2(s) = 1 + 2s + 3s^2 + 4s^3 + \cdots$$

It can be easily reconstructed from A_1.

$$A_2(s) = (A_1(s))^2 .$$

In a similar way we see that

$$A_3(s) = (A_1(s))^3$$

It is easy to calculate the coefficients of $A_3(s)$

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \;\; 63, 69, 73, 75, \;\; 75, 73, 69, 63, \;\; 55, 45, \ldots 3, 1$$

The sum of squares of all these coefficients gives the number of lucky tickets. We get 55252.

Going to complex variables

Let consider along with $A_3(s)$ polynomial the Laurent polynomial $A_3(1/s)$.

$$A_3(1/s) = a_0 + \frac{a_1}{s} + \frac{a_2}{s^2} + \ldots + \frac{a_{27}}{s^{27}}$$

The product

$$P(s) = A_3(s) A_3\left(\frac{1}{s}\right) = \left(a_0 + a_1 s + \ldots + a_{27} s^{27}\right)\left(a_0 + \frac{a_1}{s} + \frac{a_2}{s^2} + \ldots + \frac{a_{27}}{s^{27}}\right)$$

is also a Laurent polynomial.

$$P(s) = \sum_{k=-27}^{27} p_k \, s^k$$

The coefficient p_0 at s^0 in this product has the form

$$p_0 = a_0^2 + a_1^2 + \ldots + a_{27}^2$$

Now we can use the the basic fact of the theory of complex variables, Cauchy theorem, to calculate p_0

Deviation about Laurent series

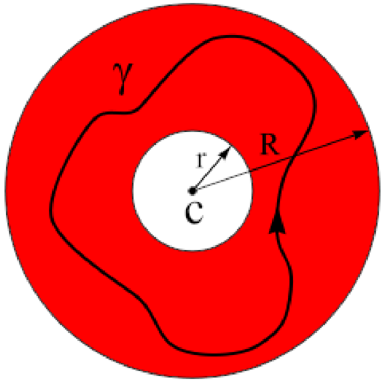
The Laurent series for a complex function $f(z)$ about a point c is given by:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n$$

where the a_n are constants, defined by a line integral

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)\,dz}{(z-c)^{n+1}}.$$

The path of integration γ is a counterclockwise closed path containing no self-intersections, enclosing c and lying in an annulus A in which $f(z)$ is holomorphic (analytic). The expansion for $f(z)$ will then be valid anywhere inside the annulus.



A Laurent series is defined with respect to a particular point c and a path of integration γ. The path of integration γ must lie in an annulus (shown here in red) inside of which $f(z)$ is holomorphic (analytic).

For any Laurent polynomial $P(s)$, its p_0 term is expressed as

$$p_0 = \frac{1}{2\pi i} \oint \frac{P(s)ds}{s},$$

where integration is over a circle including the zero.

Lucky tickets through complex variables

$$\begin{aligned} P(s) &= A_3(s)A_3(1/s) = (A_1(s))^3 \, (A_1(1/s))^3 \\ &= \left(\frac{1-s^{10}}{1-s}\right)^3 \left(\frac{1-s^{-10}}{1-s^{-1}}\right)^3 = \left(\frac{2-s^{10}-s^{-10}}{2-s-s^{-1}}\right)^3 \end{aligned}$$

Replacing $s = \exp(i\phi)$

$$\begin{aligned} p_0 &= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{2-2\cos(10\phi)}{2-2\cos\phi}\right)^3 d\phi = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\sin(5\phi)}{\sin\frac{\phi}{2}}\right)^6 d\phi \\ &= \frac{1}{\pi}\int_0^{\pi} \left(\frac{\sin(10\phi)}{\sin\phi}\right)^6 d\phi = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\sin(10\phi)}{\sin\phi}\right)^6 d\phi \end{aligned}$$

Brute force *MAPLE* calculation gives 55252. But from the point of view of physical applications it is interesting to evaluate this integral approximatively.

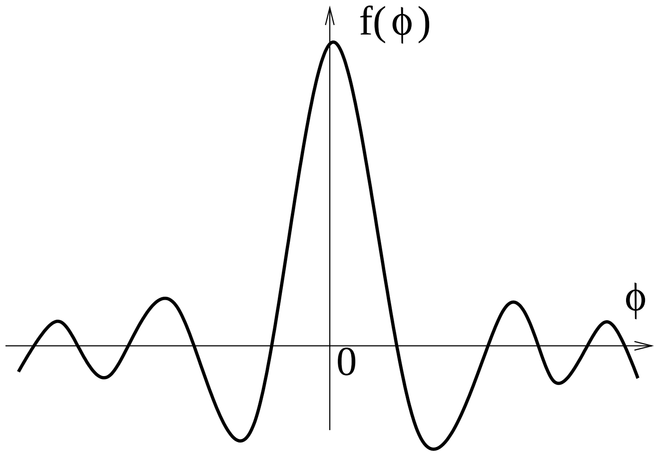


Figure 1: Oscillating function $f(\phi) = \frac{\sin(10\phi)}{\sin\phi}$

The function under integral has maximum at $\phi = 0$. Main contribution to the integral comes from the interval $[-\pi/10, \pi/10]$. At $\phi = \pm\pi/10$ the first zero of the function is located.

About stationary phase method

Method of stationary phase enables us to estimate the value of integral

$$\int_{-\pi/10}^{\pi/10} f^t d\phi = \int_{-\pi/10}^{\pi/10} e^{t \ln f} d\phi \tag{1}$$

at $t \to \infty$. The idea of the method is : At large t the value of integral is defined by the behavior of the function $\ln f$ (the phase) in the neighborhood of its stationary point 0, i.e. the point where $(\ln f)' = 0$, or equivalently $f' = 0$. Near the zero:

$$f(\phi) = \frac{\sin(10\phi)}{\sin\phi} \approx 10\left(1 - \frac{33}{2}\phi^2\right); \qquad \ln f(\phi) \approx \ln 10 - \frac{33}{2}\phi^2.$$

At large t we have

$$\int_{-\pi/10}^{\pi/10} e^{t\left(\ln 10 - \frac{33}{2}\phi^2\right)} d\phi = e^{t \ln 10} \int_{-\pi/10}^{\pi/10} e^{-\frac{33}{2} t\phi^2} d\phi \approx e^{t \ln 10} \frac{\sqrt{2\pi}}{\sqrt{33t}}$$

Here at last step we extend the limits of integration till $\pm\infty$.

Putting $t = 6$ gives for the number of lucky tickets

$$p_0 \approx \frac{10^6}{3\sqrt{11\pi}} \approx 56700.$$

The error of this approximate estimation is about 3% .

Application of inclusion - exclusion principle

Proposition. *The number of lucky tickets is equal to the number of tickets with the sum of numbers equal 27.*

Let us suppose that the ticket $a_1b_1c_1a_2b_2c_2$ is lucky.
We put in correspondence to this ticket the ticket

$$a_1b_1c_1(9-a_2)(9-b_2)(9-c_2).$$

The sum of numbers for this ticket is 27.
The correspondence
$a_1b_1c_1a_2b_2c_2 \quad \Leftrightarrow \quad a_1b_1c_1(9-a_2)(9-b_2)(9-c_2)$ is one-to-one.

This means that in order to calculate the number of lucky tickets we can calculate the number of tickets with the sum over all six positions being equal 27.

Deviation about formal logic

Let B be the set whose elements can have some of properties $c_1, \ldots . c_m$. Let $N(c_i)$, $1 \leq i \leq m$ be the number of elements of B possessing the property c_i. Let $N(c_i, c_j)$, $i \neq j$ be the number of elements of B possessing simultaneously both properties c_i and c_j, and so on.

$N(1)$ is the total number of elements in B.

Theorem. (inclusion-exclusion principle)

The number of elements in B which do not possess any of properties c_i, $i = 1, \ldots, m$ equals

$$N(1) - N(c_1) - \ldots - N(c_m) + N(c_1, c_2) + N(c_1, c_3) + \ldots - N(c_1, c_2, c_3) - \ldots$$

Symbolic writing of the same relation:

$$(1 - c_1)(1 - c_2) \ldots (1 - c_m) = 1 - c_1 - \ldots - c_m + c_1 c_2 + c_1 c_3 + \ldots - c_1 c_2 c_3 - \ldots$$

Proof

Split all elements of B into subsets $\quad B = B_0 \cup B_1 \cup \ldots \cup B_m$.

Each B_l includes all elements possessing l properties.

Let us consider the sequence of expressions:

$$N(1),$$
$$N(1) - N(c_1) - \ldots - N(c_m),$$
$$N(1) - N(c_1) - \ldots - N(c_m) + N(c_1, c_2) + \ldots + N(c_{m-1}, c_m),$$
$$\ldots\ldots$$

Now calculate how many times we take into account each element of B_l for all these expressions.

B_0	B_1	B_2	B_3	$\ldots$	B_l
1	1	1	1	$\ldots$	1
1	$1-1$	$1-2$	$1-3$	$\ldots$	$1-l$
1	$1-1$	$1-2+1$	$1-3+3$	$\ldots$	1-l+ $\binom{l}{2}$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
1	0	0	0	$\ldots$	1 - $\binom{l}{1}$ + $\binom{l}{2}$ +$\ldots$ + $(-1)^l$ $\binom{l}{l}$

We get just the number of elements possessing no properties c_i.

Lucky tickets through inclusion-exclusion principle

Let us consider all possible compositions of six non-negative numbers with the sum 27. We introduce also six properties of such compositions. Property c_i means that the number in the i-th position is not smaller than 10. In such a case, the number of lucky tickets is equal to the number of compositions which do not possess any of properties $c_1, c_2, \ldots, , c_6$.

Now we apply the theorem

$$N(1) = \left(\begin{array}{c} 32 \\ 5 \end{array} \right); \; N(c_i) = \left(\begin{array}{c} 22 \\ 5 \end{array} \right); \; N(c_i, c_j) = \left(\begin{array}{c} 12 \\ 5 \end{array} \right)$$

Note, $\quad N(c_i, c_j, c_k) = N(c_i, c_j, c_k, c_m) = \ldots = 0$

The number of lucky tickets is:

$$p_0 = \left(\begin{array}{c} 32 \\ 5 \end{array} \right) - 6 \left(\begin{array}{c} 22 \\ 5 \end{array} \right) + 15 \left(\begin{array}{c} 12 \\ 5 \end{array} \right).$$

Elementary generating functions

Simplest sequence $\quad 1, 1, 1, \ldots$.

The generating function for this sequence is

$$G(t) = 1 + t + t^2 + t^3 + t^4 + \ldots, \tag{2}$$

If we multiply (2) by t we get

$$tG(t) = t + t^2 + t^3 + t^4 + \ldots = G(t) - 1, \tag{3}$$

Consequently,

$$G(t) = \frac{1}{1-t}, \tag{4}$$

we got the sum of the geometric progression.

Newton‘s binomial theorem.

$$(1+s)^\alpha = 1 + \frac{\alpha}{1!}s + \frac{\alpha(\alpha-1)}{2!}s^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}s^3 + \ldots, \qquad (5)$$

Here α is arbitrary complex number. For α positive integer number we get standard definition. The binomial coefficient

$$\left(\begin{array}{c} N \\ k \end{array} \right) = \frac{N(N-1)\ldots(N-k+1)}{k!} \qquad (6)$$

gives the number of ways, disregarding order, that a k objects can be chosen from among N objects; more formally, the number of k-element subsets (or k-combinations) of an N-element set.

Fibonacci sequence

The Fibonacci sequence is determined by its initial two terms $f_0 = f_1 = 1$ and by the relation

$$f_{n+2} = f_{n+1} + f_n. \tag{7}$$

The initial terms of the sequence are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots, \tag{8}$$

In order to find the generating function for Fibonacci sequence

$$\mathrm{Fib}(s) = 1 + s + 2s^2 + 3s^3 + 5s^4 + \ldots \tag{9}$$

we multiply both sides of (9) by $s + s^2$.

$$\begin{aligned}
(s + s^2)\mathrm{Fib}(s) &= s + s^2 + \ 2s^3 + 3s^4 + 5s^5 + \ldots + \\
&\quad + s^2 \ + \ s^3 + 2s^4 + 3s^5 + \ldots + \\
&= s + 2s^2 + 3s^3 + 5s^4 + 8s^5 + \ldots,
\end{aligned}$$

or equivalently

$$(s+s^2)\mathrm{Fib}(s) = \mathrm{Fib}(s) - 1, \tag{10}$$

This gives the generating function

$$\mathrm{Fib}(s) = \frac{1}{1-s-s^2} \tag{11}$$

In order to find the explicit form of the coefficients we can rewrite the generating function as a sum of two elementary fractions:

$$\frac{1}{1-s-s^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{s-s_2} - \frac{1}{s-s_1}\right) = \frac{1}{\sqrt{5}}\left(\frac{1}{s_1\left(1-\frac{s}{s_1}\right)} - \frac{1}{s_2\left(1-\frac{s}{s_2}\right)}\right)$$

Here $s_1 = (-1+\sqrt{5})/2$, $s_2 = (-1-\sqrt{5})/2$ are the roots of the equation $1-s-s^2=0$. It is useful to note that $s_1 s_2 = -1$.

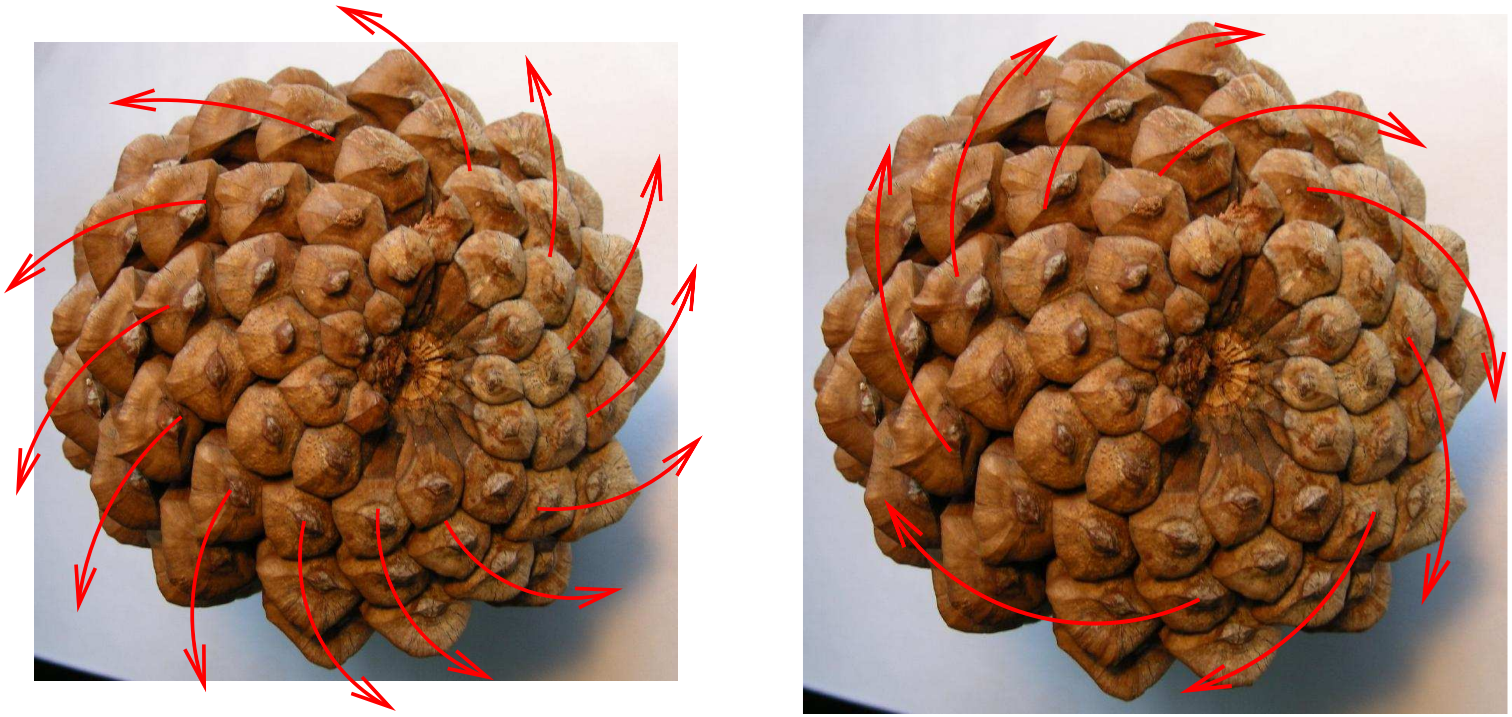
Now replacing each elementary fraction by a geometric progression

$$\mathrm{Fib}(s) = \frac{1}{\sqrt{5}s_1}\left(1 + \frac{s}{s_1} + \frac{s^2}{s_1^2} + \ldots\right) - \frac{1}{\sqrt{5}s_2}\left(1 + \frac{s}{s_2} + \frac{s^2}{s_2^2} + \ldots\right)$$

we get the explicit form of Fibonacci coefficients

$$\begin{aligned} f_n &= \frac{1}{\sqrt{5}}\left(s_1^{-1-n} - s_2^{-1-n}\right) = \frac{(-1)^n}{\sqrt{5}}\left(s_1^{n+1} - s_2^{n+1}\right) \\ &= \frac{(-1)^n}{\sqrt{5}}\left(\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}\right) \end{aligned}$$

Fibonacci numbers in Nature.



Example of spiral phyllotaxis. There are 13 left spires and 8 right spires.

Example of spiral phyllotaxis. There are three sequences of parastichies.
The number of spires in each family is 8,13,21.

Golden ratio - “Divine proportion“

Two quantities a and b are said to be in the golden ratio φ if:

$$\frac{a+b}{a} = \frac{a}{b}; \qquad \varphi = \frac{a}{b}\,.$$

This equation unambiguously defines φ.

$$\varphi^2 - \varphi - 1 = 0.$$

The only positive solution to this quadratic equation is

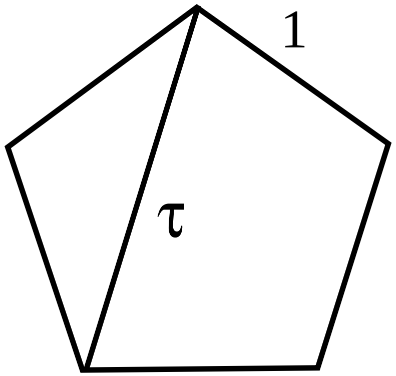
$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803\,39887\ldots$$

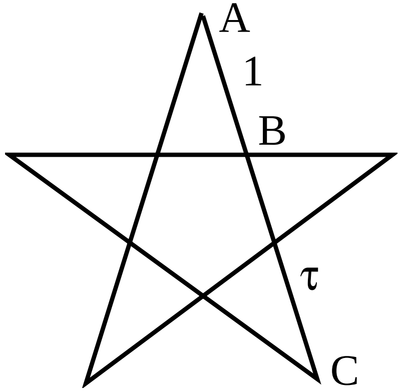
Luca Pacioli’s book *“De divina proportione“,* 1509.

“I believe that this geometric proportion served the Creator as an idea when he introduced the continuous generation of similar objects from similar objects.“ J. Kepler.

Golden ratio can be found in

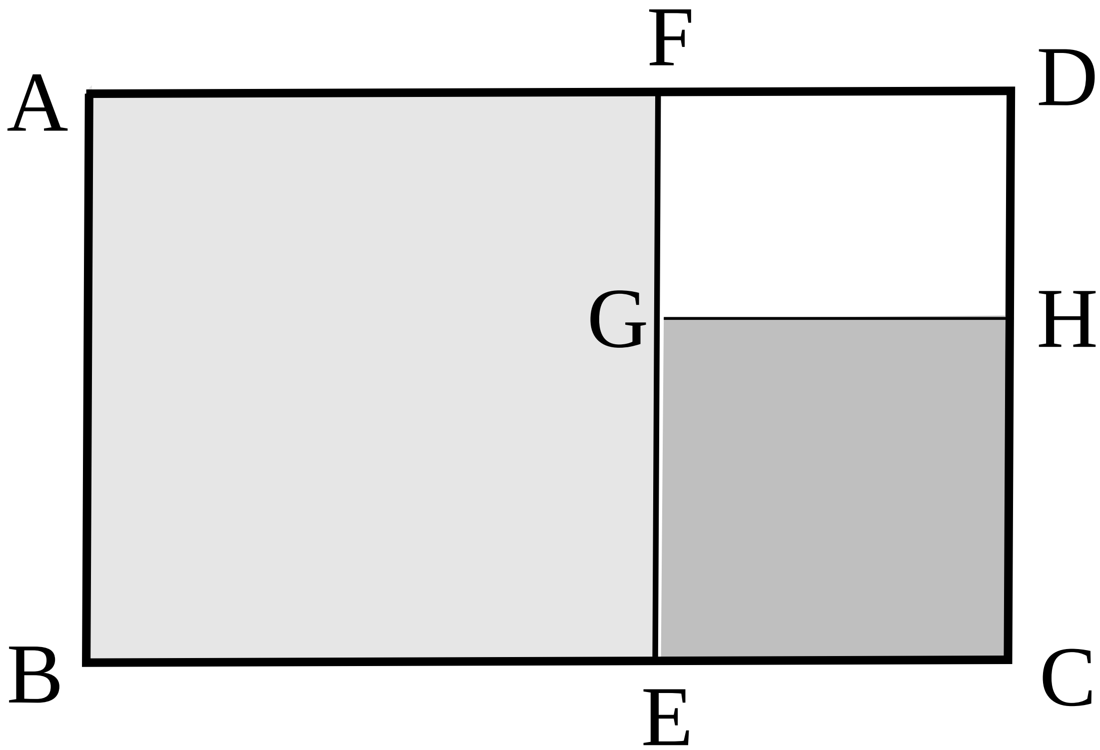
- Regular pentagon (diagonal to edge ratio; intersection of two diagonals)





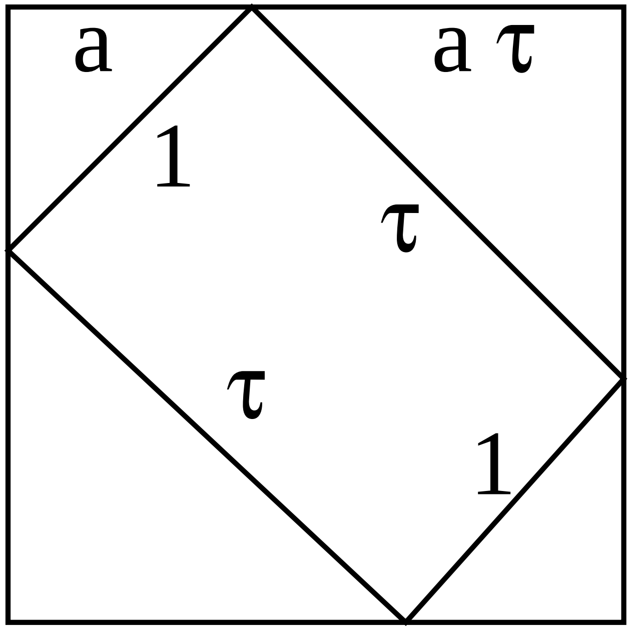
Golden ratio in pentagon and pentagram.

- Golden rectangle (after cutting a square we get a new golden rectangle)



$ABCD$ is an initial golden rectangle. After cutting out square $ABEF$ we get golden rectangle $DCEF$. After cutting out square $CEGH$ we get again the golden rectangle $DHGF$.

- Icosahedron - three golden rectangles



Three mutually orthogonal golden rectangles with common center have 12 vertexes forming regular icosahedron.
Golden rectangle inscribed into square divides its edges in golden ratio.
$\Rightarrow$ Icosahedron can be inscribed into octahedron.

Deviation: Continued fractions

Any real number x can be represented as continued fraction

$\{a_0, a_1, a_2, a_3, ...\}$, $a_0 \in Z$ et $a_i \in N,\ i \geq 1$:

$$x = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

Examples :

1. Continued fraction of rational number $\frac{10}{7}$:

$$\frac{10}{7} = 1 + \frac{3}{7} = 1 + \frac{1}{\frac{7}{3}} = 1 + \frac{1}{2 + \frac{1}{3}}.$$

2. Continued fraction for golden ratio :

$$\tau = 1 + \cfrac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}.$$

3. Continued fraction of number π :

$$\pi = 3 + \cfrac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \ldots}}}}$$

Rational approximation

The sequence $\{a_0, a_1, a_2, a_3, ...\}$ is finite for rational numbers. It is infinite for irrational numbers.

Consecutive rational approximations to number $\pi = 3.14159265358979324\ldots$:

$$3 + \frac{1}{7} \approx 3.142; \quad 3 + \frac{1}{7 + \frac{1}{15}} \approx 3.14151; \quad 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}} \approx 3.1415929;$$

$$3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292}}}} \approx 3.1415926530.$$

Rational approximation $\{a_0, a_1, a_2, ..., a_k\}$ of irrational number is better if the numbers a_i are bigger.

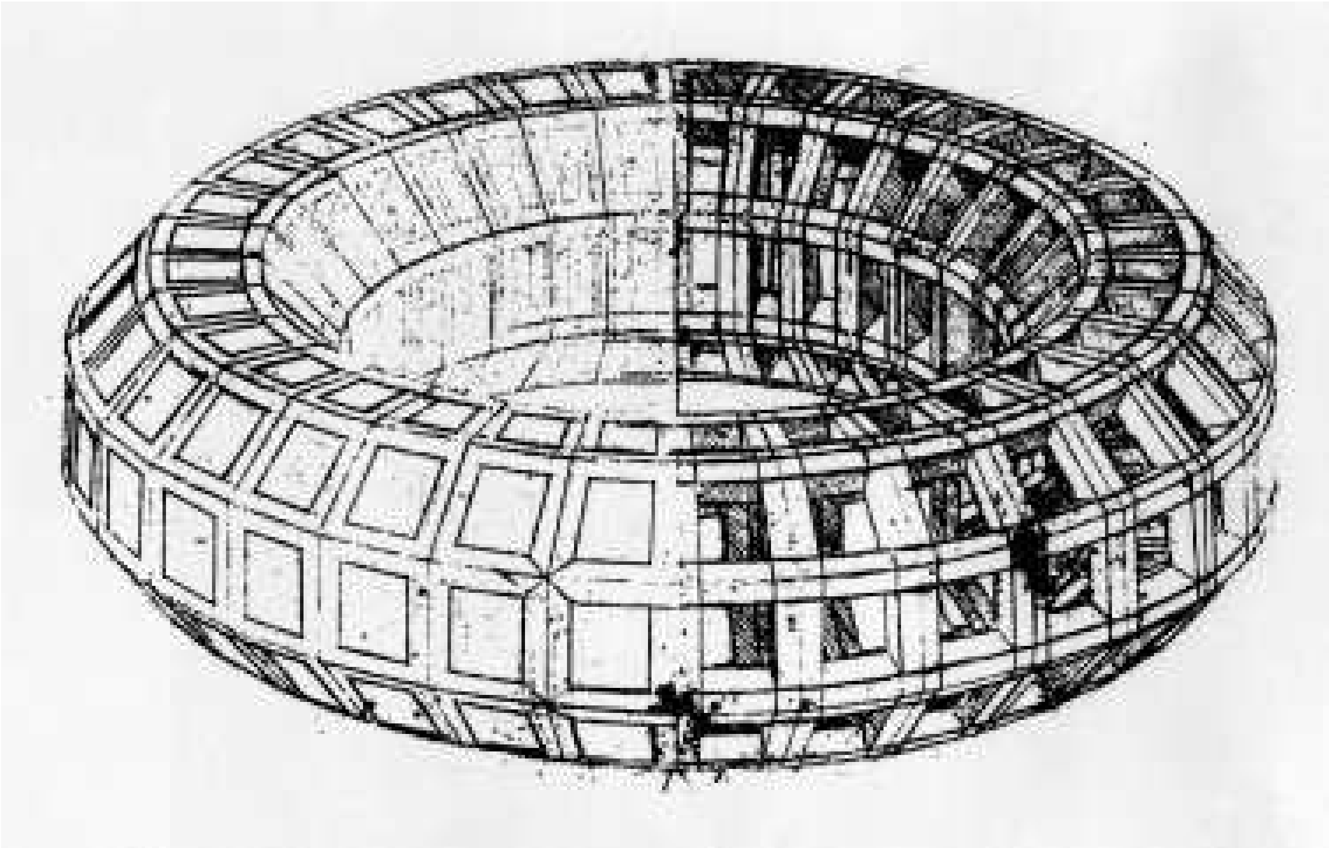
Convergence of rational approximations is the slowest for the golden ratio (all a_i are equal to 1).

Interesting question concerns the probability of appearance of numbers k (positive integer) in the representation of real numbers by continued fractions. The answer is :

$$p_k = \frac{1}{\ln 2} \ln \left(1 + \frac{1}{k(k+2)} \right) .$$

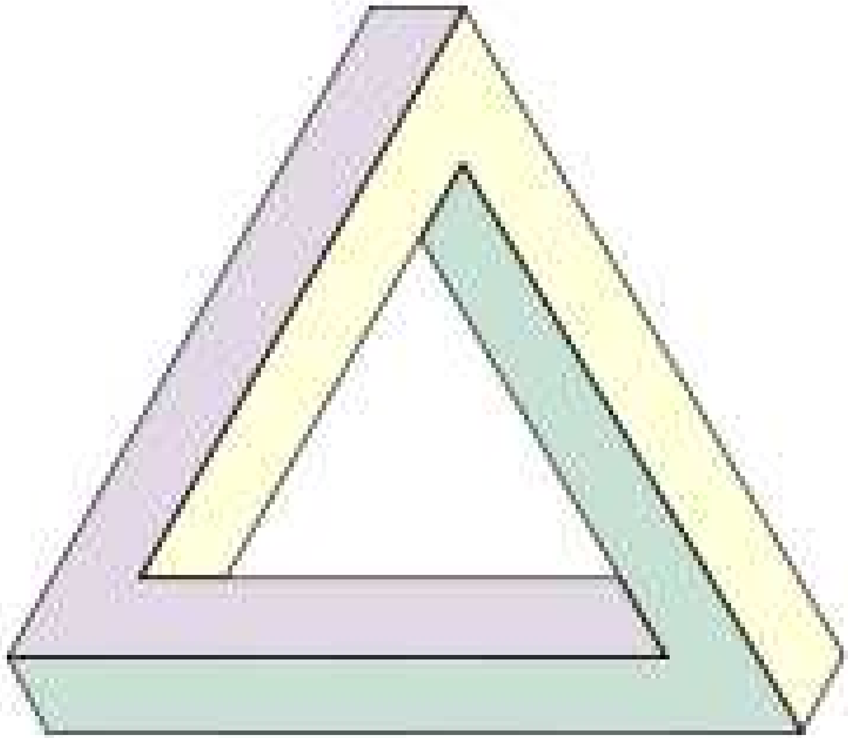
The number 1 appears the most frequently ($p_1 \approx 0.48$).

Comments to exercises



Leonardo Da Vinci (April 15, 1452 May 2, 1519)

Modern sculpture in Perth, Australia.



Penrose impossible triangle.